Mathematical Economics: Lecture 10

Yu Ren

WISE, Xiamen University

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Outline

1. Chapter 15: Implicit Functions and Their Derivatives

New Section

Chapter 15: Implicit Functions and Their Derivatives

Implicit Function

- Explicit function: $y = F(x_1, x_2, \cdots, x_n)$
- Implicit function $G(x_1, x_2, \cdots, x_n, y) = 0$

Examples

Example 15.1 the equation $4x + 2y = 5$ or $4x + 2y - 5 = 0$ express y as an implicit function of x.
write y as an explicit function of x: $y = 2.5 - 2x$

Examples

Example 15.2 Consider the equation:

$$y^2 - 5xy + 4x^2 = 0$$

convert it into an explicit function:

$$y = \frac{5x \pm \sqrt{25x^2 - 16x^2}}{2} = \frac{1}{2}(5x \pm 3x) = \begin{cases} 4x \\ x \end{cases}$$

Questions

The fact that we can write down an implicit function $G(x, y) = 0$ does not mean that this equation automatically defines y as a function of x. example: $x^2 + y^2 = 1$

Questions

two questions:

- (a) Given the implicit equation $G(x, y) = c$ and a point (x_0, y_0) such that $G(x_0, y_0) = c$, does there exist a continuous function $y = y(x)$ defined on the interval I s.t. $G(x, y(x)) = c$ for all x in I and $y(x_0) = y_0$
- (b) if $y(x)$ exists and differentiable, what is $y'(x_0)$?

Questions

two questions:

- (a) Given the implicit equation $G(x, y) = c$ and a point (x_0, y_0) such that $G(x_0, y_0) = c$, does there exist a continuous function $y = y(x)$ defined on the interval I s.t. $G(x, y(x)) = c$ for all x in I and $y(x_0) = y_0$
- (b) if $y(x)$ exists and differentiable, what is $y'(x_0)$?

Implicit Function Theorem

Theorem 15.1 Let $G(x, y)$ be a C^1 function on a ball about (x_0, y_0) in R^2. Suppose that $G(x_0, y_0) = c$ and consider the expression $G(x, y) = c$. If $(\partial G/\partial y)(x_0, y_0) \neq 0$, then there exists a C^1 function $y = y(x)$ defined on an interval I about the point x_0 s.t.

(a) $G(x, y(x)) \equiv c$ for all x in I

(b) $y(x_0) = y_0$

(c) $y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0,y_0)}{\frac{\partial G}{\partial y}(x_0,y_0)}$

Example

Example 15.7 Consider the equation

$$G(x, y) \equiv x^2 - 3xy + y^3 - 7 = 0$$

one computes that

$$\frac{\partial G}{\partial x} = 2x - 3y = -1 \text{ at(4,3)}$$
$$\frac{\partial G}{\partial y} = -3x + 3y^2 = 15 \text{ at(4,3)}$$

$$y'(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)} = \frac{1}{15}.$$

$$y_1 \approx y_0 + y'(x_0)\Delta x = 3 + (\frac{1}{15}) \cdot 3 = 3.02$$

Example

Example 15.8 the equation

$$x^2 + y^2 = 1$$

First note that

$$y'(x)|_{x=0} = -\frac{\partial G/\partial x}{\partial G/\partial y} = -\frac{2x}{2y} = -\frac{0}{2} = 0$$

an explicit formula

$$y(x) = \sqrt{1 - x^2}$$
$$y'(x) = \frac{-x}{\sqrt{1 - x^2}}$$

Higher Order derivatives & Hessian

Theorem 15.2 Let $G(x_1, \cdots, x_k, y)$ be a C^1 function on a ball about $(x_1^*, \cdots, x_k^*, y^*)$. Suppose

$$G(x_1^*, \cdots, x_k^*, y^*) = c$$

$$\frac{\partial G}{\partial y}(x_1^*, \cdots, x_k^*, y^*) \neq 0$$

Higher Order derivatives & Hessian

Theorem 15.2 Then, there is a C^1 function $y = y(x_1, \cdots, x_n)$ defined on an open ball B about $(x_1^*, \cdots, x_k^*)$

(a) $G(x_1^*, \cdots, x_k^*, y(x_1^*, \cdots, x_k^*)) \equiv c$ for all $(x_1, \cdots, x_k)$ in B

(b) $y^* = y(x_1^*, \cdots, x_k^*)$

(c) $\frac{\partial y}{\partial x_i}(x_1^*, \cdots, x_k^*) = -\frac{\frac{\partial G}{\partial x}(x_1^*, \cdots, x_k^*, y^*)}{\frac{\partial G}{\partial y}(x_1^*, \cdots, x_k^*, y^*)}$

Level curves and their tangents

Definition: A point (x_0, y_0) is called a regular point of the C^1 function $G(x, y)$ if $\frac{\partial G}{\partial x}(x_0, y_0) \neq 0$ or $\frac{\partial G}{\partial y}(x_0, y_0) \neq 0$. If every point (x, y) on the locus $G(x, y) = c$ is a regular point of G, then we call the level set a regular curve

Level curves and their tangents

Theorem 15.3 Let (x_0, y_0) be a point on the locus of points $G(x, y) = c$ in the plane, where G is a C^1 function of two variables. If $(\partial G/\partial y)(x_0, y_0) \neq 0$, then $G(x, y) = c$ defines a smooth curve around (x_0, y_0) which can be thought of as the graph of a C^1 function $y = f(x)$. Furthermore, the slope of this curve is:

$$-\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)}$$

.

Level curves and their tangents

Theorem 15.3 If $\partial G/\partial y(x_0, y_0) = 0$, but $\partial G/\partial x(x_0, y_0) \neq 0$, then the Implicit Function Theorem tells us that the locus of point $G(x, y) = c$ is a smooth curve about (x_0, y_0), which we can consider as defining x as a function of y. It also tells us that the tangent line to the curve at (x_0, y_0) is parallel to the $y-$ axis.

Level curves and their tangents

Theorem 15.4: Let G be a C^1 function on a neighborhood of (x_0, y_0). Suppose that (x_0, y_0) is a regular point of G. Then the gradient vector $\nabla G(x_0, y_0)$ is perpendicular to the level set of G at (x_0, y_0).

Level curves and their tangents

Definition: A point (x_0, y_0) is called a regular point of the C^1 function $F(x_1, \cdots, x_n)$ if $\nabla F(x^*) \neq 0$, that is, if some $(\partial F / \partial x_i)(x^*)$ is not zero. If every point (x, y) on the level set $\mathfrak{F}_c = \{(x_1, \cdots, x_n) : F(x_1, \cdots, x_n) = c\}$ is a regular point of F, then we call the level set $\mathfrak{F}_c$ a regular surface.

Level curves and their tangents

Theorem 15.6 If $F : R^n \to R^1$ is a C^1 function, if x^* is a point in R^n, and if some $(\partial F/\partial x_i)(x^*) \neq 0$ then: (a) the level set of F through x^* $\mathfrak{F}_c = \{(x_1, \cdots, x_n) : F(x_1, \cdots, x_n) = c\}$ can be viewed as the graph of a real valued C^1 function of (n-1) variables in a neighborhood of x^* (b) the gradient vector $\nabla F(x^*)$, considered as a vector at x^*, is perpendicular to the tangent hyperplane of $\mathfrak{F}_{F(x^*)}$ at x^*

Nonlinear Systems

$$
\begin{aligned}
F_1(y_1, y_2, \cdots, y_m, x_1, \cdots, x_n) &= c_1 \\
F_2(y_1, y_2, \cdots, y_m, x_1, \cdots, x_n) &= c_2 \\
\vdots &= \vdots \\
F_m(y_1, y_2, \cdots, y_m, x_1, \cdots, x_n) &= c_m
\end{aligned}
$$

Nonlinear Systems

Question:

$$\text{What is } \frac{\partial y_i}{\partial x_j}(x^*, y^*)?$$

Nonlinear Systems

$$\begin{aligned}
\frac{\partial F_1}{\partial y_1}dy_1 + \cdots \frac{\partial F_1}{\partial y_m}dy_m + \frac{\partial F_1}{\partial x_1}dx_1 + \cdots \frac{\partial F_1}{\partial x_n}dx_n &= 0 \\
\vdots &= \vdots \\
\frac{\partial F_m}{\partial y_1}dy_1 + \cdots \frac{\partial F_m}{\partial y_m}dy_m + \frac{\partial F_m}{\partial x_1}dx_1 + \cdots \frac{\partial F_m}{\partial x_n}dx_n &= 0
\end{aligned}$$

Nonlinear Systems

$$\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix} \begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} = -\begin{pmatrix} \sum_{i=1}^n \frac{\partial F_1}{\partial x_i} dx_i \\ \vdots \\ \sum_{i=1}^n \frac{\partial F_m}{\partial x_i} dx_i \end{pmatrix}$$

Nonlinear Systems

$$\begin{pmatrix} dy_1 \\ \vdots \\ dy_m \end{pmatrix} =$$

$$-\begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \cdots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}^{-1} \begin{pmatrix} \Sigma_{i=1}^{n} \frac{\partial F_1}{\partial x_i} dx_i \\ \vdots \\ \Sigma_{i=1}^{n} \frac{\partial F_m}{\partial x_i} dx_i \end{pmatrix}$$

Example

Example 15.15 Consider the system of equations

$$F_1(x, y, a) \equiv x^2 + axy + y^2 - 1 = 0$$
$$F_2(x, y, a) \equiv x^2 + y^2 - a^2 + 3 = 0$$

the Jacobian of (F_1, F_2) with respect to the endogenous variable x and y at the point x = 0, y = 1, a = 2:

Example

$$det\begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix}(0,1,2) = det\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix} = 4 \neq 0$$

we can solve the system for x and y as functions of a near (0, 1, 2)

Example

$$\frac{dy}{da} = -\frac{det\frac{\partial(F_1,F_2)}{\partial(x,a)}}{det\frac{\partial(F_1,F_2)}{\partial(x,y)}} = -\frac{det\begin{pmatrix} 2x+ay & xy \\ 2x & -2a \end{pmatrix}}{det\begin{pmatrix} 2x+ay & ax+2y \\ 2x & 2y \end{pmatrix}}$$

$$\frac{dy}{da} = -\frac{det\begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}}{det\begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}} = \frac{8}{4} = 2 > 0$$

Example

if a increases to 2.1, the corresponding y will increase to about 1.2.
Let's use another method to compute the effect on x:

$$\begin{aligned} (2x + ay)dx + (ax + 2y)dy + xyda &= 0 \\ 2xdx + 2ydy - 2ada &= 0 \end{aligned}$$

Example

plug in x = 0, y = 1, a = 2:

$$2xdx + 2ydy = 0da$$
$$0dx + 2ydy = 4da$$

so if a increases to 2.1, the corresponding x will decrease roughly to -.2.

Comparative Statics

- Economic Environment: pure exchange economy, two consumers 1 and 2, two goods x and y, initial endowments: $(e_1, 0)$, $(0, e_2)$, utility functions: U_1, U_2: $U_i(x_i, y_i) = \alpha u_i(x_i) + (1-\alpha)u_i(y_i)$, price levels: p, q.

Comparative Statics

Maximizing the utility functions, we have in equilibrium

$$\frac{\alpha}{1-\alpha}u_1'(x_1) - pu_1'(y_1) = 0$$
$$px_1 + y_1 - pe_1 = 0$$
$$\frac{\alpha}{1-\alpha}u_2'(x_2) - pu_2'(y_2) = 0$$
$$x_1 + x_2 - e_1 = 0$$
$$y_1 + y_2 - e_2 = 0$$

Comparative Statics

Question:how a change in the initial endowment e_2 affects the equilibrium consumption bundles while keeping e_1 fixed

Comparative Statics

Differentiate the equations

$$\frac{\alpha}{1-\alpha}u_1''(x_1)dx_1 - pu_1''(y_1)dy_1 - u_1'(y_1)dp = 0$$
$$pdx_1 + dy_1 - (x_1 - 1)dp = 0$$
$$\frac{\alpha}{1-\alpha}u_2''(x_2)dx_2 - pu_2''(y_2)dy_2 - u_2'(y_2)dp = 0$$
$$dx_1 + dx_2 = 0$$
$$dy_1 + dy_2 = de_2$$

Comparative Statics

Solve the above equations, we can get equation (50) and (52) in page 363