Mathematical Economics: Lecture 11

Yu Ren

WISE, Xiamen University

November 5, 2012

Outline

New Section

Chapter 16: Quadratic Forms and Definite Matrices

Quadratic Forms

- Definition: a quadratic form on R^n is a real-valued function of the form $Q(x_1, \cdots, x_n) = \sum_{i \le j} a_{ij} x_i x_j$ or $Q(X) = X'AX$
- Example: $a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$

Quadratic Forms

- Definition: a quadratic form on R^n is a real-valued function of the form $Q(x_1, \cdots, x_n) = \sum_{i \leq j} a_{ij} x_i x_j$ or $Q(X) = X'AX$
- Example: $a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2$

Definite

- positive definite: if $X^T AX > 0$ for all $X \neq 0$ in R^n
- negative definite:if $X^T AX < 0$ for all $X \neq 0$ in R^n
- positive semi-definite: if $X^T AX \geq 0$ for all $X \neq 0$ in R^n
- negative semi-definite:if $X^T AX \leq 0$ for all $X \neq 0$ in R^n

Definite

- positive definite: if $X^T AX > 0$ for all $X \neq 0$ in R^n
- negative definite:if $X^T AX < 0$ for all $X \neq 0$ in R^n
- positive semi-definite: if $X^T AX \geq 0$ for all $X \neq 0$ in R^n
- negative semi-definite:if $X^T AX \leq 0$ for all $X \neq 0$ in R^n

Definite

- positive definite: if $X^T AX > 0$ for all $X \neq 0$ in R^n
- negative definite:if $X^T AX < 0$ for all $X \neq 0$ in R^n
- positive semi-definite: if $X^T AX \geq 0$ for all $X \neq 0$ in R^n
- negative semi-definite:if $X^T AX \leq 0$ for all $X \neq 0$ in R^n

Definite

- positive definite: if $X^T AX > 0$ for all $X \neq 0$ in R^n
- negative definite:if $X^T AX < 0$ for all $X \neq 0$ in R^n
- positive semi-definite: if $X^T AX \geq 0$ for all $X \neq 0$ in R^n
- negative semi-definite:if $X^T AX \leq 0$ for all $X \neq 0$ in R^n

Submatrix

- Definition: Let A be an $n \times n$ matrix. A $k \times k$ submatrix of A formed by deleting $n-k$ columns, say columns $i_1, i_2, \cdots, i_{n-k}$ and the same $n-k$ rows $i_1, i_2, \cdots, i_{n-k}$ from A is called a k-th order principal submatrix. The determinant of a $k \times k$ principal submatrix is called a k-th order principal minor of A.

Submatrix: Example

Example 16.2 For a general 3 x 3 matrix

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

there is one third order principal minor: det(A).

Submatrix: Example

Example 16.2

there are three second order principal minors:

$$(1)\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix},$$

$$(2)\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},$$

$$(3)\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

Submatrix: Example

Example 16.2

there are three first order principal minors:
(1) $|a_{11}|$, formed by deleting the last 2 rows and columns,
(2) $|a_{22}|$, formed by deleting the first and third rows and the first and third columns,
(3) $|a_{33}|$, formed by deleting the first 2 rows and columns.

Leading Principal Minor

- Definition: Let A be an $n \times n$ matrix. The k-th order principal submatrix of A obtained by deleting the last n-k rows and the last n-k columns from A is called k-th order **leading** principal submatrix of A. Its determinant is called k-th order **leading** principal minor of A.
- Example 16.2

Leading Principal Minor

- Definition: Let A be an $n \times n$ matrix. The k-th order principal submatrix of A obtained by deleting the last n-k rows and the last n-k columns from A is called k-th order **leading** principal submatrix of A. Its determinant is called k-th order **leading** principal minor of A.
- Example 16.2

Theorems

Theorem 16.1 Let A be an $n \times n$ symmetric matrix. Then,
(a) A is positive definite if and only if all its n leading principal minors are (strictly) positive

Theorems

Theorem 16.1 Let A be an $n \times n$ symmetric matrix. Then,
(b) A is negative definite if and only if its n leading principal minors alternate in sign as follows: $|A_1| < 0$, $|A_2| > 0$, $|A_3| < 0$ etc. The kth order leading principal minor should have the same sign as $(-1)^k$.

Theorems

Theorem 16.1 Let A be an $n \times n$ symmetric matrix. Then,
(c) If some kth order leading principal minor of A is nonzero but does not fit either of the above two sign patterns, then A is indefinite.

Theorems

Theorem 16.2 Let A be an $n \times n$ symmetric matrix.
Then A is positive semidefinite if and only if every principal minor of A is ≥ 0;
A is negative semidefinite if and only if every principal minor of odd order is ≤ 0 and every principal minor of even order is ≥ 0.

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(a) if $|A_1| > 0$, $|A_2| > 0$, $|A_3| > 0$, $|A_4| > 0$, then A is positive definite (and conversely) .

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(b) if $|A_1| < 0, |A_2| > 0, |A_3| < 0, |A_4| > 0$, then A is negative definite (and conversely) .

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(c) if $|A_1| > 0, |A_2| > 0, |A_3| = 0, |A_4| < 0$, then A is indefinite because of A4 .

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.

(d) if $|A_1| < 0, |A_2| < 0, |A_3| < 0, |A_4| < 0$, then A is indefinite because of A2 (and A4) .

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(e) if $|A_1| = 0, |A_2| < 0, |A_3| > 0, |A_4| = 0$, then A is indefinite because of A2 .

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(f) if $|A_1| > 0, |A_2| = 0, |A_3| > 0, |A_4| > 0$, then A is not definite It is not negative semidefinite, but it may be positive semidefinite. To check all 15 principal minors of A. If none of the principal minors are negative, then A is positive semidefinite. If at least one of them is negative, A is indefinite.

Example

Example 16.3 Suppose A is a 4 x 4 symmetric matrix and, as usual, write $|A_i|$ for its *i*th order leading minor.
(g) if $|A_1| = 0, |A_2| > 0, |A_3| = 0, |A_4| > 0$, then A is not definite . It may be positive semidefinite or negative semidefinite. To decide, one must check all 15 principal minors of A.

Example

Example 16.4 Consider

$$A = \begin{pmatrix} 2 & 3 \\ 3 & 7 \end{pmatrix} \text{ and } B = \begin{pmatrix} 2 & 4 \\ 4 & 7 \end{pmatrix}$$

since $|A_1|$ = 2 and $|A_2|$ = 5, A is positive definite.
since $|B_1|$ = 2 and $|B_2|$ = -2, A is indefinite.

Theorems

- Definiteness of Diagonal matrix and 2×2 matrix in general form
- Section 16.3 Equation (9) (10) (11) (12) page 387

Theorems

- Definiteness of Diagonal matrix and 2×2 matrix in general form
- Section 16.3 Equation (9) (10) (11) (12) page 387

Theorems

Theorem 16.3 The quadratic form $Q(x_1, x_2) = ax_1^2 + 2bx_1x_2 + cx_2^2$ is positive (respective ly, negative) definite on the constraint set $Ax_1 + Bx_2 = 0$ if and only if $det \begin{pmatrix} 0 & A & B \\ A & a & b \\ B & b & c \end{pmatrix}$ is negative (respectively, positive).

Theorems

Theorem 16.4 To determine the definiteness of a quadratic form $Q(X) = X' A X$ when restricted to a constraint set given by m linear equations $BX = 0$, construct a symmetric matrix H as follows:

$$H = \begin{pmatrix} 0 & B \\ B^T & A \end{pmatrix}$$

check the signs of the last $n - m$ leading principal minors of H, starting with the determinant of H itself.

Theorems

- (a) If $detH$ has the same sign as $(-1)^n$ and if these last $n - m$ leading principal minors alternate in sign, then Q is negative definite on the constraint set $BX = 0$ and $X = 0$ is a strict global max of Q on this constraint set.

Theorems

- (b) If $detH$ and these last $n-m$ leading principal minors all have the same sign as $(-1)^m$, then Q is positive definite on the constraint set $BX=0$, and $X=0$ is strict global min of Q on this constraint set.

Theorems

- (c) If both of these conditions (a) and (b) are violated by nonzero leading principal minors, then Q is indefinite on the constraint set $BX = 0$.
- Remark in p390

Theorems

- (c) If both of these conditions (a) and (b) are violated by nonzero leading principal minors, then Q is indefinite on the constraint set $BX = 0$.
- Remark in p390

Example

Example 16.7 To check the definiteness of

$$Q(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 + x_3^2 + x_4^2 + 4x_2x_3 - 2x_1x_4$$

on the constraint set

$$x_2 + x_3 + x_4 = 0, \; x_1 - 9x_2 + x_4 = 0 \; ,$$

from the bordered matrix

Example

Example 16.7

$$H_6 = \left(\begin{array}{cc|cccc} 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 0 & -1 \\ 1 & -9 & 0 & -1 & 2 & 0 \\ 1 & 0 & 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Example

since the problem has n = 4 variables and m = 2 constraints, we need to check the largest n - m = 2 leading principle submatrices of H_6: H_6 itself and

$$H_5 = \left(\begin{array}{cc|ccc} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -9 & 0 \\ \hline 0 & 1 & 1 & 0 & 0 \\ 1 & -9 & 0 & -1 & 2 \\ 1 & 0 & 0 & 2 & 1 \end{array}\right)$$

Example

since m = 2 and $(-1)^2$ = +1, det$H_6 = 24 > 0$, det$H_5 = 77 > 0$, so Q is positive definite on the constraint set, and x = 0 is a min of Q restricted to the constraint set.