Mathematical Economics: Lecture 12

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Outline

New Section

Chapter 17: Unconstrained Optimization

Definitions

- **max**: $F(x^*) \geq F(x)$ for all $x \in U$
- **strict max**: $F(x^*) > F(x)$ for all $x \in U \neq x^*$
- **local max**: $F(x^*) \geq F(x)$ for all $x \in B_r(x^*) \cap U$
- **strict local max**: $F(x^*) > F(x)$ for all $x \in B_r(x^*) \cap U \neq x^*$
- Critical Point: $DF(x^*) = 0$

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Theorem 17.1

Theorem 17.1 (first order conditions): x^* is a local max (or min), interior of U, then $\frac{\partial F}{\partial x_i}(x^*) = 0$ for $i = 1, \cdots, n$

Example

Example 17.1 To find the local maxs and mins of $F(x, y) = x^3 - y^3 + 9xy$

$$\frac{\partial F}{\partial x} = 3x^2 + 9y = 0 \text{ and } \frac{\partial F}{\partial y} = -3y^2 + 9x = 0$$

solutions are two points (0, 0) (3, -3). At this stage, we can conclude that the only candidates for a max or min of F are these two points. We are unable to say whether either of these two is a max or min.

Second Order Conditions

Hessian of F : $D^2F(x^*) = (\frac{\partial^2 F}{\partial x_i \partial x_j})_{ij}$

Second Order Conditions

Theorem 17.2 (second order conditions): x^* is a critic point of F,
(a) if $D^2F(x^*)$ is negative definite, x^* is strict local max;
(b) if $D^2F(x^*)$ is positive definite, x^* is strict local min;
(c) if $D^2F(x^*)$ is indefinite, x^* is called saddle point: x^* is min in some direction, and max in some other direction

Second Order Conditions

Example of saddle point :

$$F(x_1, x_2) = x_1^2 - x_2^2$$

Theorems

Theorem 17.6 : x^* is an interior and local max, then $DF(x^*) = 0$ and $D^2F(x^*)$ is negative **semi**definite

Theorems

Theorem 17.7: Let $F : U \to R^1$ be a C^2 function of n variables. Suppose that x^* is an interior point of U.
(a) If x^* is a local min of F, then $(\partial F / \partial x_i)(x^*) = 0$ for $i = 1, \cdots, n$ and all the principal minors of the Hessian $D^2 F(x^*)$ are ≥ 0.
(b) If x^* is a local max of F, then

Example

Example 17.2

In example 17.1, we computed the critical points of $F(x, y) = x^3 - y^3 + 9xy$ are (0, 0) and (3, -3), by differentiating the first derivatives, we compute that the Hessian of F is

$$\begin{pmatrix} F_{xx} & F_{yx} \\ F_{yx} & F_{yy} \end{pmatrix} = \begin{pmatrix} \text{6x} & 9 \\ 9 & \text{-6y} \end{pmatrix}$$

Example

Example 17.2

The first order leading principle minor is $F_{xx} = 6x$ and the second order leading principle minor is detD^2F(x)= -36xy-81. At (0, 0), these two minors are 0 and -81, respectively. Since the second order leading principle minor is negative, (0, 0) is a saddle of F $-$ neither a max nor a min. At (3, -3), these two minors are 18 and 243. Since these two numbers are positive, D^2F(3, -3) is positive definite and (3, -3) is a strict local min of F.

Example

Example 17.2

Notice that (3, -3) is not a global min, because at the point (0, n), F(0, n)= $-n^3$, which goes to $-\infty$ as n $\to \infty$

Theorems

- Global maximum minimum
- **Theorem 17.8**: If $D^2F(x)$ is negative semidefinite for all x and $DF(x^*) = 0$, then x^* is a global max
- **Theorem 17.8**: If $D^2F(x)$ is positive semidefinite for all x and $DF(x^*) = 0$, then x^* is a global min

Theorems

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Example

Example 17.3 A monopolist producing a single product has two types of customers. If it produces Q_1 units for customers of type 1, then these customers are willing to pay a price of 50-5Q_1 dollars per unit. If it produces Q_2 units for customers of type 2, then these customers are willing to pay a price of 100-10Q_2 dollars per unit. The monopolist's cost of manufacturing Q units of output is 90+20Q dollars. In order to maximize profits, how much should the monopolist produce for each market?

Example

Example 17.3

This discriminating monopolist's profit function is

$$\begin{aligned} F(Q_1, Q_2) &= Q_1(50 - 5Q_1) + Q_2(100 - 10Q_2) \\ &- (90 + 20(Q_1 + Q_2)). \end{aligned}$$

Example

The critical point of F satisfy:

$$\frac{\partial F}{\partial Q_1} = 50 - 10Q_1 - 20 = 0, \text{ or } Q_1 = 3$$
$$\frac{\partial F}{\partial Q_2} = 50 - 10Q_2 - 20 = 0, \text{ or } Q_2 = 4$$

Now check the second order conditions. Since

$$F_{Q_1 Q_1} = -10, \; F_{Q_2 Q_2} = -20, F_{Q_1 Q_2} = F_{Q_2 Q_1} = 0,$$

the first order leading principle minor of D^2F(3, 4) is -10 and second order leading principle minor is 200. Therefore, F is a concave function and the point (3, 4) is the profit-maximizing