Mathematical Economics: Lecture 13

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Outline

New Section

Chapter 18: Constrained Optimization I

Definitions

- Objective function: $f(x_1, x_2, \cdots, x_n)$
- Constraint function:
  $g_k(x_1, \cdots, x_n) \leq b_k$
  $f_k(x_1, \cdots, x_n) = c_k$

Example 18.1

Example 18.1 (Utility Maximization Problem) In this most basic problem, X_i represents the amount of commodity i and $f(x_1, \ldots, x_n)$, usually written as $U(x_1, \ldots, x_n)$, measures the individual's level of utility or satisfaction with consuming x_1 units of good 1, x_2 units of good 2, and so on. Let $p_1, \ldots, p_n$ denote the prices of the commodities and let I denote the individual's income.

Example 18.1

The consumer wants to

$$\begin{aligned} \text{maximize} \quad & U(x_1, \ldots, x_n) \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 + \cdots + p_n x_n \leq I \end{aligned}$$

To be consistent with the general format in (1), the nonnegativity constraints $x_i \geq 0$ should be written as $-x_i \leq 0$ so that all inequality constraints are written with $\leq$ signs.

Example 18.2

Example 18.2 (Utility Maximization with Labor/Leisure Choice) Let U, $x_1, \ldots, x_n$, $p_1, \ldots, p_n$ be as in the preceding example. In addition, let w denote the wage rate, I' the consumer's nonwage income, l_0 hours of labor, and l_1 hours of leisure. The consumer has $I' + wl_0$ dollars to spend and wants to

$$\begin{array}{ll} \text{maximize} & U(x_1, \ldots, x_n, l_1) \\ \text{subject to} & p_1x_1 + p_2x_2 + \cdots + p_nx_n \leq I' + wl_0, \\ & l_0 + l_1 = 24, \\ & x_1 \geq 0, \ldots, x_n \geq 0, l_0 \geq 0, l_1 \geq 0 \end{array}$$

Example 18.3

Example 18.3 (Profit Maximization of a Competitive Firm) Suppose that a firm in a competitive industry uses n inputs to manufacture its product. Let y denote the amount of its output, and let $x_1, \ldots, x_n$ denote the amounts of its inputs $-$ all flow concepts. Let $y = f(x_1, \ldots, x_n)$ denote the firm's production function, describing the maximal amount of output that can be produced from bundle $(x_1, \ldots, x_n)$. Let p be the unit price of the output and let w_i denote the cost of input i.

Example 18.3

The firm's goal is to choose $(x_1, \dots, x_n)$ to maximize its profit

$$\Pi(x_1, \dots, x_n) = pf(x_1, \dots, x_n) - \sum_1^n w_i x_i$$

Example 18.3

under the constraints

$$pf(x_1, \ldots, x_n) - \sum_{1}^{n} w_i x_i \geq 0,$$
$$g_1(x) \leq b_1, \ldots, g_k(x) \leq b_k,$$
$$x_1 \geq 0, \ldots, x_n \geq 0.$$

The first inequality constraints reflects the requirement that the firm make a nonnegative profit. The g_j-constraints represent constraints on the availability of the inputs.

Equality Constraints

- Two variables and one equality constraint
  Setup:
  $$\max f(x_1, x_2)$$
  s.t.
  $$h(x_1, x_2) = c$$
- (2) (3) (4) (5) (6) in Page 414
- Figure 18.1

NDCQ

- NonDegenerate Constraint Qualification: the rank of Jacobian $Dh(x^*)$ is equal to the number of the constraints, then x^* satisfies NDCQ.
- x^* is called a critical point of $h = (h_1, h_2, \cdots, h_m)$ if the rank of the matrix $Dh(x^*)$ is less than m.

Equality Constraints

Solution (**Theorem 18.1**):Suppose (x_1^*, x_2^*) is not a critical point of h. Then there is a real number μ^* such that (x_1^*, x_2^*, μ^*)

$$L(x_1, x_2, \mu) \equiv f(x_1, x_2) - \mu(h(x_1, x_2) - c)$$

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x) - \mu \frac{\partial h}{\partial x_1}(x) &= 0 \\ \frac{\partial f}{\partial x_2}(x) - \mu \frac{\partial h}{\partial x_2}(x) &= 0 \\ h(x_1, x_2) - c &= 0 \end{aligned}$$

μ is called a Lagrangian multiplier

Example 18.4

Example 18.4 Let's use Theorem 18.1 to solve a simple utility maximization problem:

$$\begin{array}{ll} \text{maximize} & f(x_1, x_2) = x_1 x_2 \\ \text{subject to} & h(x_1, x_2) \equiv x_1 + 4x_2 = 16 \end{array}$$

Since the gradient of h is (1, 4), h has no critical points and the constraint qualification is satisfied. From the Lagrangian

$$L(x_1, x_2, \mu) = x_1 x_2 - \mu(x_1 + 4x_2 - 16),$$

Example 18.4

and set its partial derivatives equal to zero:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= x_2 - \mu = 0 \\
\frac{\partial L}{\partial x_2} &= x_1 - 4\mu = 0 \\
\frac{\partial L}{\partial \mu} &= -(x_1 + 4x_2 - 16) = 0
\end{aligned}$$

we conclude the solution of this system is

$$x_1 = 8,\ x_2 = 2, \mu = 2.$$

Theorem 18.1 states that the only candidate for a solution is $x_1 = 8, x_2 = 2$.

Example 18.5

Example 18.5 Let's work out a more complex example:

$$\begin{array}{ll} \text{maximize} & f(x_1, x_2) = x_1^2 x_2 \\ \text{subject to} & C_h = \{(x_1, x_2) : 2x_1^2 + x_2^2 = 3\}. \end{array}$$

To check the constraint qualification, we compute the critical points of $h(x_1, x_2) = 2x_1^2 + x_2^2$. The only such critical point at (x_1, x_2)=(0, 0) $-$ a point which is not in the constraint set C_h.

Example 18.5

Now, from the Lagrangian

$$L(x_1, x_2, \mu) = x_1^2 x_2 - \mu(2x_1^2 + x_2^2 - 3),$$

compute its partial derivatives, and set them equal to 0:

$$\begin{aligned}
\frac{\partial L}{\partial x_1} &= 2x_1 x_2 - 4\mu x_1 = 2x_1(x_2 - 2\mu) = 0 \\
\frac{\partial L}{\partial x_2} &= x_1^2 - 2\mu x_2 = 0 \\
\frac{\partial L}{\partial \mu} &= -2x_1^2 - x_2^2 + 3 = 0
\end{aligned}$$

Example 18.5

The first equation yields $x_1 = 0$ or $x_2 = 2\mu$.
If $x_1 = 0$, therefore $(0, \sqrt{3}, 0)$ and $(0, -\sqrt{3}, 0)$ are two solutions of the system.
If $x_1 \neq 0$, then $x_2 = 2\mu$, then
$x_1^2 = x_2^2 \Rightarrow x_1 = \pm 1, x_2 = \pm 1$.
If $x_2 = +1$, and $\mu = 0.5$. If $x_2 = -1$, and $\mu = -0.5$. Then we obtain four more solutions of the system

$(1, 1, 0.5)$, (-1, -1, -0.5), $(1, -1, -0.5)$, (-1, 1, 0.5) .

Example 18.5

Since

$$f(1,1) = f(-1,1) = 1,$$
$$f(1,-1) = f(-1,-1) = -1,$$
$$f(0,\sqrt{3}) = f(0,-\sqrt{3}) = 0.$$

the max occurs at (1, 1) and (-1, 1). Note that (1, -1) and (-1, -1) minimize f on C_h.

Equality Constraints

Several equality constraints
Setup:

$$\max f(x_1, x_2, \cdots, x_n)$$

s.t.

$$h_1(x) = a_1; \cdots, h_m(x) = a_m$$

Equality Constraints

Solution (**Theorem 18.2**): Lagrangian Function: NDCQ

$$
\begin{aligned}
L(x,\mu) &\equiv f(x) - \mu_1(h_1(x) - a_1) - \mu_2(h_2(x) - a_2) \\
&- \cdots - \mu_m(h_m(x) - a_m)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial x_1}(x^*,\mu^*) = 0, \quad \cdots \quad , \frac{\partial L}{\partial x_n}(x^*,\mu^*) = 0 \\
\frac{\partial L}{\partial \mu_1}(x^*,\mu^*) = 0, \quad \cdots \quad , \frac{\partial L}{\partial \mu_n}(x^*,\mu^*) = 0
\end{aligned}
$$

Example 18.6

Example 18.6 Consider the problem:

$$\begin{aligned} \text{maximize} \quad & f(x, y, z) = xyz \\ \text{subject to} \quad & h_1(x, y, z) = x^2 + y^2 = 1 \\ & \text{ and } h_2(x, y, z) = x + z = 1. \end{aligned}$$

First compute the Jacobian matrix of the constraint functions

$$Dh(x, y, z) = \begin{pmatrix} 2x & 2y & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Example 18.6

Since any point with x=y=0 would violate the first constraint, all points in the constraint set satisfy NDCQ. Next from the Lagrangian

$$L(x, y, z, \mu_1, \mu_2) = xyz - \mu_1(x^2 + y^2 - 1) - \mu_2(x + z - 1),$$

and set its first partial derivatives equal to 0:

Example 18.6

$$
\begin{aligned}
\frac{\partial L}{\partial x} &= yz - 2\mu_1 x - \mu_2 = 0 \\
\frac{\partial L}{\partial y} &= xz - 2\mu_1 y = 0 \\
\frac{\partial L}{\partial z} &= xy - \mu_2 = 0 \\
\frac{\partial L}{\partial \mu_2} &= 1 - x^2 - y^2 = 0 \\
\frac{\partial L}{\partial \mu_1} &= 1 - x - z = 0
\end{aligned}
$$

Example 18.6

Solve the second and third equations for μ_1 and μ_2 in terms of x, y, and z and plug these into the first equation to obtain

$$y^2 z - x^2 z - xy^2 = 0$$

Then, solve the fourth equation for y^2 in terms of x^2 and the last equation for z in terms of x, then

$$(1 - x^2)(1 - x) - x^2(1 - x) - x(1 - x^2) = 0$$

Example 18.6

so $x = \frac{1}{6}(-1 \pm \sqrt{13})$, approximately -0.7676 and 0.4343. We obtain the four solution candidates

$$\begin{aligned} x \simeq 0.4343, \quad & y \simeq \pm 0.9008, \quad z \simeq 0.5657; \\ x \simeq -.07676, \quad & y \simeq \pm 0.6409, \quad z \simeq 1.7676; \end{aligned}$$

Evaluate the objective functions at these four points, we find that the maximizer is

$$x \simeq -.07676, \; y \simeq -0.6409, z \simeq 1.7676$$

Inequality Constraints

- Figure 18.4 Figure 18.5
- One inequality constraint:
  Setup:

$$\max f(x, y)$$

s.t.

$$g(x, y) \leq b$$

Inequality Constraints

Solution (**Theorem 18.3**): If $\frac{\partial g}{\partial x}(x^*, y^*) \neq 0$ or $\frac{\partial g}{\partial y}(x^*, y^*) \neq 0$ Lagrangian Function:

$$L(x, y, \lambda) \equiv f(x, y) - \lambda[g(x, y) - b]$$

$$\begin{aligned} \frac{\partial L}{\partial x}(x^*, y^*, \lambda^*) &= 0 \\ \frac{\partial L}{\partial y}(x^*, y^*, \lambda^*) &= 0 \\ \lambda^*[g(x^*, y^*) - b] &= 0 \\ \lambda^* &\geq 0 \\ g(x^*, y^*) &\leq b \end{aligned}$$

Example 18.7

Example 18.7 Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x,y) = xy \\ \text{subject to} & g(x,y) = x^2 + y^2 \leq 1 \end{array}$$

The only critical point of g occurs at the origin – far away from the boundary of the constraint set $x^2 + y^2 = 1$. So the constraint qualification will be satisfied any candidate for a solution. From the lagrangian

$$L(x,y,\lambda) = xy - \lambda(x^2 + y^2 - 1),$$

and write out the first order conditions described in Theorem 18.3:

Example 18.7

$$\frac{\partial L}{\partial x} = y - 2\lambda x = 0, \;\; \frac{\partial L}{\partial y} = x - 2\lambda y = 0,$$
$$\lambda(x^2 + y^2 - 1) = 0, \;\; x^2 + y^2 \leq 1 \;, \;\; \lambda \geq 0$$

Example 18.7

The first two equations yield

$$\lambda = \frac{y}{2x} = \frac{x}{2y} \text{ or } x^2 = y^2.$$

If $\lambda = 0$, then $x = y = 0$, which is a candidate for a solution.
If $\lambda \neq 0$, then $x^2 + y^2 - 1 = 0 \Rightarrow x^2 = y^2 = \frac{1}{2} \Rightarrow$ $x = \pm\frac{1}{\sqrt{2}}, x = \pm\frac{1}{\sqrt{2}}$, then we find the following four candidates:

Example 18.7

$$x = +\frac{1}{\sqrt{2}}, \quad y = +\frac{1}{\sqrt{2}}, \quad \lambda = +\frac{1}{2}$$
$$x = -\frac{1}{\sqrt{2}}, \quad y = -\frac{1}{\sqrt{2}}, \quad \lambda = +\frac{1}{2}$$
$$x = +\frac{1}{\sqrt{2}}, \quad y = -\frac{1}{\sqrt{2}}, \quad \lambda = -\frac{1}{2}$$
$$x = -\frac{1}{\sqrt{2}}, \quad y = +\frac{1}{\sqrt{2}}, \quad \lambda = -\frac{1}{2}$$

Example 18.7

We disregard the last two candidates since they involve a negative multiplier. Plugging the left three candidates into the object function, we find that

$$x = \frac{1}{\sqrt{2}}, \ y = \frac{1}{\sqrt{2}} \ \text{and} \ x = -\frac{1}{\sqrt{2}}, \ y = -\frac{1}{\sqrt{2}}$$

are the solutions of our original problem. The two points with the negative multipliers are the solutions of the problem of minimizing xy on the constraint set $x^2 + y^2 \leq 1$.

Inequality Constraints

Several Inequality Constraint:
Setup:

$$\max f(x)$$

$s.t.$

$$g_1(x) \leq b_1, \cdots, g_k(x) \leq b_k$$

Inequality Constraints

Solution (**Theorem 18.4**): NDCQ condition
Assume the first k_0 constraints are binding at x^* and the last $k - k_0$ are not binding. NDCQ holds if the rank of

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(x^*) \end{pmatrix}$$

is k_0

Inequality Constraints

Solution (**Theorem 18.4**): Lagrangian Function:

$$L(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_k) \equiv$$
$$f(x) - \lambda_1[g_1(x) - b_1] - \cdots - \lambda_1[g_k(x) - b_k]$$

$$\frac{\partial L}{\partial x_1}(x^*, \lambda^*) = 0, \quad \cdots \quad , \frac{\partial L}{\partial x_n}(x^*, \lambda^*) = 0$$
$$\lambda_1^*[g_1(x^*, y^*) - b_1] = 0 \quad \cdots \quad \lambda_k^*[g_k(x^*, y^*) - b_k] = 0$$
$$\lambda_1^* \geq 0 \quad \cdots \quad \lambda_k^* \geq 0$$
$$g_1(x^*) \leq b_1 \quad \cdots \quad g_k(x^*) \leq b_k$$

Example 18.8

Example 18.8 Consider again the standard utility maximization problem of Example 18.1. We continue to ignore the nonnegativity constraints but do not force the budget constraint to be binding in the statement of the problem.

$$\begin{aligned} \text{maximize} \quad & U(x_1, x_2) \\ \text{subject to} \quad & p_1 x_1 + p_2 x_2 \leq I \end{aligned}$$

Example 18.8

We assume that for each commodity bundle (x_1, x_2),

$$x = \frac{\partial U}{\partial x_1}(x_1, x_2) > 0 \text{ or } x = \frac{\partial U}{\partial x_2}(x_1, x_2) > 0$$

This is a version of the usual monotonicity or nonsatiation assumption. It states that the commodities under study are *goods* in that increasing consumption increases utility. Since the usual constraint qualification is satisfied, so from the lagrangian

$$L(x_1, x_2, \lambda) = U(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - I)$$

Example 18.8

and compute its x_1- and x_2- critical points:

$$\begin{aligned}\frac{\partial L}{\partial x_1}(x_1, x_2) &= \frac{\partial U}{\partial x_1}(x_1, x_2) - \lambda p_1 = 0,\\ \frac{\partial L}{\partial x_2}(x_1, x_2) &= \frac{\partial U}{\partial x_1}(x_1, x_2) - \lambda p_2 = 0.\end{aligned}$$

Example 18.8

At the maximizer, the multiplier λ cannot be zero; otherwise both $\frac{\partial U}{\partial x_1}$ and $\frac{\partial U}{\partial x_2}$ would be zero $-$ a contradiction to our monotonicity assumption. Since

$$\lambda > 0 \text{ and } \lambda(p_1 x_1 + p_2 x_2 - I) = 0,$$

it follows that $p_1 x_1 + p_2 x_2 = I$; the consumer will spend all available income and we can treat the budget constraint as an equality constraint.

Example 18.9

Example 18.9 Consider the problem:

$$\begin{aligned} \max \quad & f(x,y,z) = xyz \\ s.t \quad & x+y+z \leq 1, x \geq 0, y \geq 0, z \geq 0 \end{aligned}$$

Example 18.9

we rewrite the three nonnegatitivity constraints as

$$-x \leq 0, \; -y \leq 0, \; \text{and} \; -z \leq 0.$$

Example 18.9

The Jacobian of the constraint functions is

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Example 18.9

Since the NDCQ holds at any solution candidate. From the lagrangian

$$L(x,y,z,\lambda_1,\lambda_2,\lambda_3,\lambda_4) = xyz - \lambda_1(x+y+z-1) \\ -\lambda_2(-x) - \lambda_3(-y) - \lambda_4(-z).$$

we can rewrite it more aesthetically as

$$\begin{aligned} L(x,y,z,\lambda_1,\lambda_2,\lambda_3,\lambda_4) &= xyz - \lambda_1(x+y+z-1) \\ &+ \lambda_2 x + \lambda_3 y + \lambda_4 z. \end{aligned}$$

Example 18.9

according to Theorem 18.4:

$$
\begin{aligned}
&(1) \quad \frac{\partial L}{\partial x} = yz - \lambda_1 + \lambda_2 = 0, \\
&(2) \quad \frac{\partial L}{\partial y} = xz - \lambda_1 + \lambda_3 = 0, \\
&(3) \quad \frac{\partial L}{\partial z} = xy - \lambda_1 + \lambda_4 = 0,
\end{aligned}
$$

Example 18.9

$$
\begin{array}{llll}
(4) & \lambda_1(x+y+z-1)=0, & (5) & \lambda_2 x = 0, \\
(6) & \lambda_3 y = 0, & (7) & \lambda_4 z = 0, \\
(8) & \lambda_1 \geq 0, & (9) & \lambda_2 \geq 0, \\
(10) & \lambda_3 \geq 0, & (11) & \lambda_4 \geq 0, \\
(12) & x+y+z-1 \leq 1, & (13) & x \geq 0, \\
(14) & y \geq 0, & (15) & z \geq 0.
\end{array}
$$

Example 18.9

Rewrite conditions 1, 2, and 3, without minus signs, as

$$\lambda_1 = yz + \lambda_2 = xz + \lambda_3 = xy + \lambda_4$$

If $\lambda_1 = 0$, then $yz = xz = xy = 0$ and $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0$
If $\lambda_1 > 0$, suppose $x = 0$, then $\lambda_1 = \lambda_3 = \lambda_4 > 0 \Rightarrow y = z = 0-$ a contradiction to $x + y + z = 1$, so $x > 0$. Similarly $y > 0, z > 0 \Rightarrow \lambda_2 = \lambda_3 = \lambda_4 = 0$ and $yz = xz = xy = \frac{1}{3}$. Since $f(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) = \frac{1}{27} > 0$, is the solution of the constraint maximization problem.

Mixed Constraints

- Setup:

$$\max f(x)$$

s.t.

$$g_1(x) \leq b_1, \quad \cdots \quad , g_k(x) \leq b_k$$
$$h_1(x) = c_1, \quad \cdots \quad , h_m(x) = c_m$$

Mixed Constraints

Solution (**Theorem 18.5**): NDCQ condition
Assume the first k_0 constraints are binding at x^* and the last $k - k_0$ are not binding.

$$r\begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x^*) & \cdots & \frac{\partial g_1}{\partial x_n}(x^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial g_{k_0}}{\partial x_1}(x^*) & \cdots & \frac{\partial g_{k_0}}{\partial x_n}(x^*) \\ \frac{\partial h_1}{\partial x_1}(x^*) & \cdots & \frac{\partial h_1}{\partial x_n}(x^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial h_m}{\partial x_1}(x^*) & \cdots & \frac{\partial h_m}{\partial x_n}(x^*) \end{pmatrix} = ?$$

Mixed Constraints

Solution (**Theorem 18.5**): Lagrangian Function:

$$
\begin{aligned}
L(x_1, \cdots, x_n, \lambda_1, \cdots, \lambda_k, \mu_1, \cdots, \mu_m) &\equiv \\
f(x) - \lambda_1[g_1(x) - b_1] - \cdots - \lambda_k[g_k(x) &- b_k] \\
-\mu_1[h_1(x) - c_1] - \cdots - \mu_m[h_m(x) &- c_m]
\end{aligned}
$$

Mixed Constraints

$$
\begin{aligned}
\frac{\partial L}{\partial x_1}(x^*, \lambda^*, \mu^*) = 0 \quad &\cdots \quad \frac{\partial L}{\partial x_n}(x^*, \lambda^*, \mu^*) = 0 \\
\lambda_1^*[g_1(x^*, y^*) - b_1] = 0 \quad &\cdots \quad \lambda_k^*[g_k(x^*, y^*) - b_k] = 0 \\
h_1(x^*) = c_1 \quad &\cdots \quad h_m(x^*) = c_m \\
\lambda_1^* \geq 0 \quad &\cdots \quad \lambda_k^* \geq 0 \\
g_1(x^*) \leq b_1 \quad &\cdots \quad g_k(x^*) \leq b_k
\end{aligned}
$$

Example 18.10

Example 18.10 Consider the problem:

$$\begin{aligned} \text{maximize} \quad & x - y^2 \\ \text{subject to} \quad & x^2 + y^2 = 4, \ \ x \geq 0, \ \ y \geq 0. \end{aligned}$$

Checking the NDCQ, first note that the gradient of $x^2 + y^2$ is zero only at the origin, a point which is not in the constraint set.

Example 18.10

If either nonnegativity constraint is binding, then the solution candidates is (2, 0) and (0, 2). In both cases, the corresponding 2X2 Jacobian matrix of constraints has rank two. Therefore, the NDCQ will automatically be satisfied. From the lagrangian

$$L = x - y^2 - \mu(x^2 + y^2 - 4) + \lambda_1 x + \lambda_2 y.$$

The first order conditions become:

Example 18.10

$$
\begin{array}{ll}
(1) & \frac{\partial L}{\partial x} = 1 - 2\mu x + \lambda_1 = 0, \\
(2) & \frac{\partial L}{\partial y} = -2y - 2\mu y + \lambda_2 = 0, \\
(3) & x^2 + y^2 - 4 = 0,
\end{array}
$$

$$
\begin{array}{ll}
(4) & \lambda_1 x = 0, \\
(6) & \lambda_1 \geq 0, \\
(8) & x \geq 0
\end{array}
$$

$$
\begin{array}{ll}
(5) & \lambda_2 y = 0, \\
(7) & \lambda_2 \geq 0, \\
(9) & y \geq 0.
\end{array}
$$

Example 18.10

Write 1 as $1 + \lambda_1 = 2\mu x$.Since $\lambda_1 \geq 0, 1 + \lambda_1 > 0$. Therefore, $\mu > 0, x > 0, \lambda_1 = 0$. Write 2 as $2y(1 + \mu) = \lambda_2$. Since $1 + \lambda_1 > 0$, either both y and λ_2 are zero or both are positive. By 5, both cannot be positive. Therefore, $\lambda_2 = y = 0$. Now, $x = 2$ by 3 and 8, $\lambda_1 = 0$ by 4, and $\mu = 1/4$ by 1. This leads to the solution

$$(x, y, \mu, \lambda_1, \lambda_2) = (2, 0, 1/4, 0, 0).$$

Example 18.11

Example 18.11 Consider the problem:

$$\begin{array}{ll} \text{minimize} & f(x,y) = 2y - x^2 \\ \text{subject to} & x^2 + y^2 \leq 1, \;\; x \geq 0, \;\; y \geq 0. \end{array}$$

From the lagrangian

$$\begin{aligned} L(x, y, \lambda_1, \lambda_2, \lambda_3) &= 2y - x^2 \\ &- \lambda_1(-x^2 - y^2 + 1) - \lambda_2 x - \lambda_3 y. \end{aligned}$$

Example 18.11

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= -2x + 2\lambda_1 x - \lambda_2 = 0, \\ \frac{\partial L}{\partial y} &= 2 + 2\lambda_1 y - \lambda_3 = 0, \\ \lambda_1(-x^2 - y^2 + 1) &= 0, \\ \lambda_2 x &= 0, \\ \lambda_3 y &= 0 \\ \lambda_1, \lambda_2, \lambda_3 &\geq 0, \end{aligned}$$

Example 18.11

rewrite the $\frac{\partial L}{\partial x} = 0$ equations, then $2x + \lambda_2 = 2\lambda_1 x$, $2 + 2\lambda_1 y = \lambda_3$. Since $\lambda_3 \geq 2 > 0$, we conclude that $y = 0$ and $\lambda_3 = 2$.
Next, if $x + 0 \Rightarrow \lambda_1 = 0, y = 0$ and $\lambda_2 = 0$, thus $f(0,0) = 0$.
If $x > 0 \Rightarrow \lambda_2 = 0, \lambda_1 = 1, y = 0, x = 1$, thus $f(1,0) = -1$.
We conclude that $(x,y) = (1,0)$ minimizes $f(x,y) = 2y - x^2$ on the constraint set.

Example 18.12

Example 18.12 In this framework, the Kuhn-Tucker Lagrangian for the usual utility maximization problem of Example 18.1 would be:

$$\widetilde{L}(x_1, x_2, \lambda) = U(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - I),$$

Example 18.12

The first order conditions are:

$$\frac{\partial U}{\partial x_1} - \lambda p_1 \leq 0, \qquad \frac{\partial U}{\partial x_2} - \lambda p_2 \leq 0,$$

$$x_1\left(\frac{\partial U}{\partial x_1} - \lambda p_1\right) = 0, \qquad x_2\left(\frac{\partial U}{\partial x_2} - \lambda p_2\right) = 0,$$

$$\frac{\partial \widetilde{L}}{\partial \lambda} = I - p_1 x_1 - p_2 x_2 \geq 0,$$

$$\lambda \frac{\partial \widetilde{L}}{\partial \lambda} = \lambda(I - p_1 x_1 - p_2 x_2) = 0$$

Example 18.13

Example 18.13 Consider the problem:

$$\begin{aligned} \text{minimize} \quad & f(x,y) = x^2 + x + 4y^2 \\ \text{subject to} \quad & 2x + 2y \leq 1, \;\; x \geq 0, \;\; y \geq 0. \end{aligned}$$

The Jacobian of the constraint functions is

$$\begin{pmatrix} 2 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}$$

Example 18.13

At most two constraints can be binding at the same time, and any 2X2 submatrix of the Jacobian has rank two. Therefore, the NDCQ will hold at any solution candidate. From the lagrangian

$$\begin{aligned} L(x, y, \lambda_1, \lambda_2, \lambda_3) &= x^2 + x + 4y^2 \\ &- \lambda_1(2x + 2y - 1) + \lambda_2 x + \lambda_3 y. \end{aligned}$$

Example 18.13

The first order conditions are:

$$\frac{\partial L}{\partial x} = -2x + 1 - 2\lambda_1 + \lambda_2 = 0,$$

$$\frac{\partial L}{\partial y} = 8y - 2\lambda_1 + \lambda_3 = 0,$$

$$\lambda_1(2x + 2y - 1) = 0,\ \lambda_2 x = 0,\ \lambda_3 y = 0,$$

$$\lambda_1 \geq 0,\ \lambda_2 \geq 0\ ,\lambda_3 \geq 0,$$

$$2x + 2y \leq 1,\ x \geq 0,\ y \geq 0.$$

Example 18.13

Since $2\lambda_1 \geq 1 > 0$ implies that $2x + 2y = 1$ is binding.
Look at the first case
$\lambda_2 > 0 \Rightarrow x = 0, y = 0.5, \lambda_3 = 0, \lambda_1 = 2, \lambda_2 = 3.$
So the assumption $\lambda_2 > 0$ leads to the candidate

$$(x, y, \lambda_1, \lambda_2, \lambda_3) = (0, 0.5, 2, 3, 0),$$

try the opposite case: $\lambda_2 = 0$, then we find
$2x + 1 + \lambda_2 = 2\lambda_1$ and $2 = 10y + \lambda_3$.

Example 18.13

So this leads to the conclusion that either $y = 0$ or $\lambda_3 = 0$, and we get two candidates: $(x, y, \lambda_1, \lambda_2, \lambda_3) = (0.5, 0, 1, 0, 2)$ or $(x, y, \lambda_1, \lambda_2, \lambda_3) = (0.3, 0.2, 0.8, 0, 0)$ by evaluating the objective function at each of these three candidates, we find that the constrained maximum occurs at the point $x = 0, y = 0.5$, where $\lambda_1 = 2, \lambda_2 = 3$ and $\lambda_3 = 0$.