Mathematical Economics: Lecture 14

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Outline

1 Chapter 19: Constrained Optimization II

New Section

Chapter 19: Constrained Optimization II

The meaning of the multiplier

One Equality constraint:

$$\begin{aligned} \max \quad & f(x, y) \\ \text{s.t.} \quad & h(x, y) = a \end{aligned}$$

The meaning of the multiplier

Theorem 19.1 Let f and h be C^1 functions of two variables. For any fixed value of the parameter a, let $(x^*(a), y^*(a))$ be the solution of problem (1) with corresponding multiplier $\mu^*(a)$. Suppose that x^*, y^* and μ^* are C^1 functions of a and that NDCQ holds at $x^*(a), y^*(a), \mu^*(a)$. Then

$$\mu^*(a) = \frac{d}{da} f(x^*(a), y^*(a))$$

Example 19.1

Example 19.1 In example 18.5, we found that a maximizer of $f(x_1, x_2) = x_1^2 x_2$ on the constraint set $2x_1^2 + x_2^2 = 3$ is $x_1 = 1, x_2 = 1$, with multiplier $\mu = 0.5$. The maximum value of f is $f^* = f(1, 1) = 1$. Redo the problem, this time using constraint $2x_1^2 + x_2^2 = 3.3$. The same computation as in Example 18.5 yields the solution $x_1 = x_2 = \sqrt{1.1}$, with maximum value $f^* = (1.1)^{3/2} \approx 1.1537$, an increase of 0.1537 over the original f^*.

Example 19.1

On the other hand, Theorem 19.1 predicts that changing the right-hand side of the constraint by 0.3 unit would change the maximum value of the object function by roughly

$$0.3 \cdot \mu = 0.3 \cdot 0.5 = 0.15 \text{ units},$$

an approximation correct to two decimal places.

Several Equality Constraints

Theorem 19.2 Let $f, h_1, \cdots, h_m$ be C^1 functions on R^n. Let $a = (a_1, \cdots, a_m)$ be an $m-$tuple of exogenous parameters, and consider the problem (P_a) of maximizing $f(x_1, x_2, \cdots, x_n)$ subject to the constraints $h_1(x_1, \cdots, x_n) = a_1$, $\cdots$ $h_m(x_1, \cdots, x_n) = a_m$. Let $x_1^*(a), \cdots, x_n^*(a)$ denote the solution of problem (P_a) with corresponding Lagrange multiplier $\mu_1^*(a), \cdots, \mu_m^*(a)$.

Theorem 19.2

Suppose further that the x_i^* and μ_j^* are differentiable functions of $(a_1, \cdots, a_m)$ and that NDCQ holds. Then, for each $j = 1, \cdots, m$

$$\mu_j^*(a_1 \cdots a_m) = \frac{\partial}{\partial a_j} f(x_1^*(a_1 \cdots a_m) \cdots x_n^*(a_1 \cdots a_m))$$

Inequality constraints

Theorem 19.3 Let $a^* = (a_1^*, \cdots, a_k^*)$ be a $k-$tuple. Consider the problem (Q_a) of maximizing $f(x_1, \cdots, x_n)$ subject to the k inequality constraints $g_1(x_1, \cdots, x_n) \leq a_1^*, \cdots,$ $g_k(x_1, \cdots, x_n) \leq a_k^*$. Let $x_1^*(a), \cdots, x_n^*(a)$ denote the solution of problem (Q_a) with corresponding Lagrange multiplier $\lambda_1^*(a), \cdots, \lambda_k^*(a)$. Suppose further that the x_i^* and μ_j^* are differentiable functions of $(a_1, \cdots, a_m)$ and that NDCQ holds. Then, for each $j = 1, \cdots, m$

$$\lambda_j^*(a_1^*, \cdots, a_k^*) = \frac{\partial}{\partial a_j} f(x_1^*, \cdots, x_n^*)$$

Inequality constraints

- Explain the case where $\lambda = 0$
- Interpreting the Multiplier: $\lambda_j^*(a)$ internal value , shadow price

Example 19.2

Example 19.2 In example 18.9, we computed that the max of xyz on the set

$$x + y + z \leq 1,\ x \geq 0, y \geq 0\ z \geq 0$$

occurs at $x = y = z = 1/3$, where $xyz = 1/27$. The four multipliers are 1/9, 0, 0, and 0, respectively.

Example 19.2

(a)If we change the first constraint to $x + y + z \leq 0.9$, we compute that the solution occurs at $x = y = z = 0.3$, where $xyz = 0.027$. Theorem 19.3 predicts that the new optimal value would be

$$\frac{1}{27} + \frac{1}{9} \cdot (-\frac{1}{10}) \approx 0.0259,$$

an estimate that is off by only .0011 or four percent.

Example 19.2

(b)If, instead, we change the second constraint form $x \geq 0$ to $x \geq 0.1$, we do not change the solution or the optimum value because the new region is a subset of the old region and it still contains the optimal point for the old region. This result is consistent with Theorem 19.3 since the multiplier for the (nonbinding) constraint $x \geq 0$ was zero.

Envelope Theorems

Theorem 19.4 Let $f(x;a)$ be a C^1 function of $x \in R^n$ and the scalar a. For each choice of the parameter a, consider the unconstrained maximization problem: max $f(x;a)$ with respect to x. Let $x^*(a)$ be a solution of this problem. Suppose that $x^*(a)$ is a C^1 function of a. Then

$$\frac{d}{da} f(x^*(a), a) = \frac{\partial}{\partial a} f(x^*(a), a)$$

Example 19.3

Example 19.3 Consider the problem of maximizing

$$f(x, a) = -a^3x^4 + 15x^3 - e^ax^2 + 17$$

around $a = 1$. Since f is a quartic polynomial in x with a negative leading coefficient when $f(x) \to -\infty$ as $x \to \pm\infty$. Therefore, f does have a finite global maximizer $x^*(a)$ for each value of a near 1.

Example 19.3

Example 19.3 By (10),

$$\frac{d}{da}f(x^*(a);a) = \frac{\partial}{\partial a}f(x^*(a);a) = -3a^2x^{*4} - e^a x^{*2},$$

which is negative at all a and all $x \neq 0$. So, even without solving for the optimal $x^*(a)$, we can tell that as a increases beyond 0, $f(x^*(a);a)$ is a decreasing function of a. The peak of the graph of the function $x \to f(x;a)$ decreases as a increases.

Example 19.4

Example 19.4 What will be the effect of a unit increase in a on the maximum value of $f(x;a) = -x^2 + 2ax + 4a^2$, where we maximize f with respect to x for each a? Since

$$f'(x) = -2x + 2a = 0$$

so, $x^*(a) = a$. Then $f(x^*(a);a) = f(a,a) = -a^2 + 2a \cdot a + 4a^2 = 5a^2$, which will increase at a rate of $10a$ as a increases.

Example 19.4

Example 19.4 If, instead, we had applied the envelope theorem,

$$\frac{df^*}{da} = \frac{\partial f^*}{\partial a}(x^*(a); a) = 2x + 8a = 10a,$$

since $x^*(a) = a$.

Example 19.5

Example 19.5 A silicon Valley firm produces an output of microchips denoted by y and has a cost function $c(y)$, with $c'(y) > 0$ and $c''(y) > 0$.Of the chips it produces, a fraction $1 - \alpha$ are unavoidably defective and cannot be sold. Working chips can be sold at price p, and the microchip market is highly competitive. How will an increase in production quality affect the firm's profit?

The firm's profit function is

$$\pi(p, \alpha) = \max_{y}[p\alpha y - c(y)],$$

Example 19.5

The conditions on the cost function guarantee that there is a nonzero profit-maximizing output $y^*(\alpha)$ which depends smoothly on α. The derivative of optimal profit π with respect to α is

$$\frac{d\pi}{da} = \frac{\partial}{\partial a}(p\alpha y - c(y)) = py > 0.$$

increasing the fraction of nondefective chips will increase the firm's profit. Once again, we were able to determine this without actually solving for the optimal output.

Constrained Problems

Theorem 19.5 Let $f, h_1, \cdots, h_k$: $R^n \times R^1 \to R^1$ be C^1 functions. Let $x^*(a) = (x_1^*(a), \cdots, x_n^*(a))$ denote the solution of the problem of maximizing $x \to f(x; a)$ on the constraint set $h_1(x; a) = 0, \cdots, h_k(x; a) = 0$ for any fixed choice of the parameter a. Suppose that $x^*(a)$ and the Lagrange multiplier $\mu_1^*(a), \cdots, \mu_m^*(a)$. Suppose further that the x_i^* and μ_j^* are differentiable functions of $(a_1, \cdots, a_m)$ and that NDCQ holds.

Constrained Problems

Theorem 19.5 Then, for each $j = 1, \cdots, m$

$$\frac{d}{da} f(x^*(a); a) = \frac{\partial L}{\partial a}(x^*(a), \mu(a); a)$$

where L is the natural Lagrange for this problem.

Example 19.6

Example 19.6 Change the constraint in Example 18.7 from $x^2 + y^2 \leq 1$ to $x^2 + 1.1y^2 \leq 1$, keeping the objective function $f(x, y) = xy$. If we write both constraints as $x^2 + ay^2 \leq 1$, the Lagrangian for the parameterized problem is

$$L(x, y, \lambda; a) = xy - \lambda(x^2 + ay^2 - 1)$$

Example 19.6

the solution for the original ($a = 1$) problem was $x = y = 1/\sqrt{2}, \lambda = 1/2$. The envelope Theorem tells us that as a changes from 1 to 1.1, the optimal value of f changes by approximately

$$\frac{\partial L}{\partial a}(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2}; 1) \cdot (0.1).$$

Example 19.6

since $\frac{\partial L}{\partial a} = -\lambda y^2 = -\frac{1}{2} \cdot (\frac{1}{\sqrt{2}})^2 = -\frac{1}{4}$, the optimal value will decrease by approximately .1/4=.025 to .475. One can calculate directly that the solution to the new problem is $x = \frac{1}{\sqrt{2}}, y = \frac{1}{\sqrt{2.2}}$, with maximum objective value of f approximately equal to 0.4767.

Second Order Conditions

Theorem 19.7 Let f and h be C^2 functions on R^2. Consider the problem of maximizing f on the constraint set $C_h = \{(x, y) : h(x, y) = c\}$. Form the Lagrangian

$$L(x, y, \mu) = f(x, y) - \mu(h(x, y) - c).$$

Second Order Conditions

Theorem 19.7 Suppose that (x^*, y^*, μ^*) satisfies

- $\frac{\partial L}{\partial x} = 0$, $\frac{\partial L}{\partial y} = 0$, $\frac{\partial L}{\partial \mu} = 0$ at (x^*, y^*, μ^*)
- $det \begin{pmatrix} 0 & \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial^2 L}{\partial x^2} & \frac{\partial^2 L}{\partial x \partial y} \\ \frac{\partial h}{\partial y} & \frac{\partial^2 L}{\partial x \partial y} & \frac{\partial^2 L}{\partial y^2} \end{pmatrix} > 0$

Then (x^*, y^*) is a local maximizer of f

Second Order Conditions

Theorem 19.6 Let f, $h_1, \cdots, h_k$ be C^2 functions on R^n. Consider the problem of maximizing f on the constraint set $C_h = \{x : h_1(x) = c_1, \cdots, h_k(x) = c_k\}$. Form the Lagrangian and suppose that

Second Order Conditions

Theorem 19.6

- x^* lies in the constraint set C_h
- there exist $\mu_1^*, \cdots, \mu_k^*$ such that $\frac{\partial L}{\partial x_1} = 0, \cdots \frac{\partial L}{\partial x_n} = 0, \frac{\partial L}{\partial \mu_1} = 0, \cdots, \frac{\partial L}{\partial \mu_k} = 0$ at $(x_1^*, \cdots, x_n^*, \mu_1^*, \cdots, \mu_k^*)$.
- the Hessian of L with respect to x at (x^*, μ^*) is negative definite on the linear constraint set $\{v : Dh(x^*)v = 0\}$.

Then x^* is a strict local constrained max of f on C_h.

Second Order Conditions

mixed constraints: Theorem 19.8

Let f, $g_1, \cdots, g_m$, $h_1, \cdots, h_k$ be C^2 functions on R^n. Consider the problem of maximizing f on the constraint set

$$\begin{aligned} C_{g,h} &= \{x : g_1(x) \leq b_1, \cdots, g_m(x) \leq b_m, \\ & \quad h_1(x) = c_1, \cdots, h_k(x) = c_k\} \end{aligned}$$

Form the Lagrangian

Second Order Conditions

Theorem 19.8

- Suppose that there exist $\lambda_1^*, \cdots, \lambda_m^*, \mu_1^*, \cdots, \mu_k^*$ such that the first order conditions are satisfied.
- Suppose only $g_1, \cdots, g_e$ are binding at x^*. Write $(g_1, \cdots, g_e)$ as g_E. Suppose that the Hessian of L with respect to x at (x^*, λ^*, μ^*) is negative definite on the linear constraint set

$$\{V : Dg_E(x^*)V = 0 \text{ and } Dh(x^*)V = 0\}$$

Then x^* is a strict local constrained max of f.

Example 19.7

Example 19.7 In Example 18.5, we consider the problem

$$\begin{array}{ll} \text{maximize} & f(x_1, x_2) = x_1^2 x_2 \\ \text{subject to} & C_h = \{(x_1, x_2) : 2x_1^2 + x_2^2 = 3\}. \end{array}$$

There, we have six solutions to the first order conditions

$$(x_1, x_2, \mu) = \begin{cases} (0, \pm\sqrt{3}, 0) \\ (\pm 1, +1, +0.5) \\ (\pm 1, -1, -0.5) \end{cases}$$

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Example 19.7

Let's use the second order conditions to decide which of these points are local maxima and which are local minima. The Hessian

$$H = \begin{pmatrix} 0 & h_{x_1} & h_{x_2} \\ h_{x_1} & L_{x_1x_1} & L_{x_1x_2} \\ h_{x_2} & L_{x_2x_1} & L_{x_2x_2} \end{pmatrix} = \begin{pmatrix} 0 & 4x_1 & 2x_2 \\ 4x_1 & 2x_2 - 4\mu & 2x_1 \\ 2x_2 & 2x_1 & -2\mu \end{pmatrix}$$

Example 19.7

This problem has $n = 2$ variables and $k = 1$ equality constraints. As Theorem 19.7 indicates, we need only check the sign of $n - k = 1$ determinant $-$ the determinant of H itself.
At the points $(\pm 1, -1, -0.5)$,

$$H = \begin{pmatrix} 0 & \pm 4 & 2 \\ \pm 4 & 0 & \pm 2 \\ -2 & \pm 2 & 1 \end{pmatrix}$$

In either case, det$H$=-16; so these two points are local minima.

Example 19.7

At the points $(\pm 1, 1, 0.5)$,

$$H = \begin{pmatrix} 0 & \pm 4 & 2 \\ \pm 4 & 0 & \pm 2 \\ -2 & \pm 2 & -1 \end{pmatrix}$$

In either case, det$H$=+48; so these two points are local maxima.

Example 19.7

When $\mu = 0$, the corresponding bordered Hessian is

$$H = \begin{pmatrix} 0 & 0 & \pm 2\sqrt{3} \\ 0 & \pm 2\sqrt{3} & 0 \\ \pm 2\sqrt{3} & 0 & 0 \end{pmatrix}$$

For $(x_1, x_2) = (0, +\sqrt{3})$, det$H$=$-24\sqrt{3} < 0$, this point is a local min.
For $(x_1, x_2) = (0, -\sqrt{3})$, det$H$=$+24\sqrt{3} > 0$, this point is a local max.

Example 19.8

Example 19.8 Consider the problem:

$$\begin{array}{ll} \text{maximize} & f(x, y, z) = x^2y^2z^2 \\ \text{subject to} & x^2 + y^2 + z^2 = 3 \end{array}$$

The first order conditions are:

$$\begin{aligned} \frac{\partial L}{\partial x} &= 2xy^2z^2 - 2\mu x = 0, \\ \frac{\partial L}{\partial y} &= 2x^2yz^2 - 2\mu y = 0, \\ \frac{\partial L}{\partial z} &= 2x^2y^2z - 2\mu z = 0, \\ -\frac{\partial L}{\partial \mu} &= x^2 + y^2 + z^2 - 3 = 0, \end{aligned}$$

with solution $x^2 = y^2 = z^2 = \mu = 1$.

Example 19.8

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with solution $x^2 = y^2 = z^2 = \mu = 1$.

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with solution $x^2 = y^2 = z^2 = \mu = 1$.

Example 19.8

The bordered Hessian for this problem is

$$\begin{pmatrix} 0 & 2x & 2y & 2z \\ 2x & 2y^2z^2 - 2\mu & 4xyz^2 & 4xy^2z \\ 2y & 4xyz^2 & 2x^2z^2 - 2\mu & 4x^2yz \\ 2z & 4xy^2z & 4x^2yz & 2x^2y^2 - 2\mu \end{pmatrix}$$

Example 19.8

At $x = y = z = \mu = 1$, the bordered Hessian becomes

$$\begin{pmatrix} 0 & 2 & 2 & | & 2 \\ 2 & 0 & 4 & | & 4 \\ 2 & 4 & 0 & | & 4 \\ - & - & - & - & \\ 2 & 4 & 4 & & 0 \end{pmatrix}$$

Since $n = 3$ and $k = 1$, we have to check the the signs of the two leading principal minors: $\det H_3 = 32$ and $\det H_4 = -192$, the candidate $x = y = z = 1$ is local constrained max by Theorem 19.6.