Mathematical Economics: Lecture 15

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Outline

New Section

Chapter 20: Homogeneous and Homothetic Functions

Definitions

- Definition: For any scalar k, a real-valued function $f(x_1, x_2, \cdots, x_n)$ is homogenous of degree k if $f(tx_1, \cdots, tx_n) = t^k f(x_1, \cdots, x_n)$ for all $x_1, \cdots, x_n$ and all $t > 0$
- focus on homogenous functions defined on the positive orthant R^n_+

Example

Example 20.1

$$
\begin{aligned}
&(a) \quad x_1^2 x_2 + 3x_1 x_2^2 + x_2^3 \\
&(b) \quad x_1^7 x_2 x_3^2 + 5x_1^6 x_2^4 - x_2^5 x_3^5 \\
&(c) \quad 4x_1^2 x_2^3 - 5x_1 x_2^2 \\
&(d) \quad z = a_1 x_1 + a_2 x_2 + \cdots a_n x_n \\
&(e) \quad z = \sum a_{ij} x_i x_j
\end{aligned}
$$

Example 20.2

Example 20.2 Replace x_1, x_2, x_3 by tx_1, tx_2, tx_3 respectively in Example 20.1a and 20.1b yields

$$\begin{aligned} & (tx_1)^2(tx_2) + 3(tx_1)(tx_2)^2 + (tx_2)^3 \\ = \; & t^2x_1^2tx_2 + 3tx_1t^2x_2^2 + t^3x_2^3 \\ = \; & t^3(x_1^2x_2 + 3x_1x_2^2 + x_2^3) \end{aligned}$$

Example 20.2

$$(tx_1)^7(tx_2)(tx_3)^2 + 5(tx_1)^6(tx_2)^4 - (tx_2)^5(tx_3)^5$$
$$= t^{10}(x_1^7 x_2 x_3^2 + 5x_1^6 x_2^4 - x_2^5 x_3^5).$$

However, no such relationship exists for Example 20.1*c*.

Example 20.3

Example 20.3 The function

$$f_1(x_1, x_2) = 30x_1^{1/2}x_2^{3/2} - 2x_1^3x_2^{-1}$$

is homogeneous of degree two. The function

$$f_2(x_1, x_2) = x_1^{1/2}x_2^{1/4} + x_1^2x_2^{-5/4}$$

is homogeneous of degree three-quarters. The fractional exponents in these two examples give one reason for making the restriction $t > 0$ in the definition of homogeneous.

Example 20.3

Example 20.3 The function

$$f_3(x_1, x_2) = \frac{x_1^7 - 3x_1^2 x_2^5}{x_1^4 + 2x_1^2 x_2^2 + x_2^4}$$

is homogeneous of degree three($= 7 - 4$).

Example 20.4

- **Example 20.4** However, the only homogeneous functions of one variable are the functions of the form $z = ax^k$, where k is any real number.

  To prove this statement, let $z = f(x)$ be an arbitrary homogeneous function of one variable. Let $a \equiv f(1)$ and let x be arbitrary. Then,

$$f(x) = f(x \cdot 1) = x^k f(1) = ax^k.$$

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Definitions

- Homogenous function in Economics:
  constant returns to scale $\leftrightarrow$ homogenous of degree one;
  increasing returns to scale $\leftrightarrow$ $k > 1$;
  decreasing returns to scale $\leftrightarrow$ $k < 1$
  Cobb-Douglas function: $q = Ax_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$

Properties

Theorem 20.1 Let $z = f(x)$ be a C^1 function on an open cone in R^n. If f is homogeneous of degree k, its first order partial derivatives are homogeneous of degree $k - 1$.

Properties

Theorem 20.2 Let $q = f(x)$ be a C^1 homogeneous function on the positive orthant. The tangent planes to the level sets of f have constant slope along each ray from the origin. Figure 20.2 and Figure 20.3 (income expansion path)

Properties

Theorem 20.3 Let $U(x)$ be a utility function on R_+^n that is homogeneous of degree k. Then, (i) the MRS is constant along rays from the origin. (ii) income expansion paths are rays from the origin. (iii) the corresponding demand depends linearly on income (iv) the income elasticity of demand is identically 1.

Properties

Let $q = f(x)$ be a production function on R^n, that is homogeneous of degree k. Then (i) the marginal rate of technical substitution (MRTS) is constant along rays from the origin (ii) the corresponding cost function is homogeneous of degree $1/k$: $C(q) = bq^{1/k}$

Calculus criterion

Theorem 20.4 Let $f(x)$ be a C^1 homogeneous function of degree k on R^n_+. Then $X^T \nabla f(x) = kf(x)$

Calculus criterion

Theorem 20.5 Suppose that $f(x_1, \cdots, x_n)$ is a C^1 function on the positive orthant R^n_+. Suppose that $X^T \nabla f(x) = kf(x)$ for all $X \in R^n_+$. Then f is homogeneous of degree k.

Properties

- Homogenizing a function
- **Theorem 20.6**: Let f be a real-valued function defined on a cone C in R^n. Let k be an integer. Define a new function $F(x_1, x_2, \cdots, x_m, z) = z^k f(\frac{x_1}{z}, \frac{x_2}{z}. \cdots, \frac{x_n}{z})$. Then F is a homogeneous function of degree k. And $F(x, 1) = f(x)$.

Properties

Theorem 20.7: Suppose $(x, z) \rightarrow f(x, z)$ is a function that is homogenous of degree k on a set $C \times R_+$ for some cone C in R^n and that $F(x, 1) = f(x)$ for all x.
Then$F(x_1, x_2, \cdots, x_m, z) = z^k f(\frac{x_1}{z}, \frac{x_2}{z}. \cdots, \frac{x_n}{z})$

Example 20.5

Example 20.5 If $f(x) = x^a$ on R_+, then its homogenization of degree one is

$$\begin{aligned} F(x,y) &= y \cdot \left(\frac{x}{y}\right)^a \\ &= x^a y^{1-a} \end{aligned}$$

Example 20.6

- **Example 20.6** If f is the nonhomogeneous function $x \to x - ax^2$, then its homogenization of degree one is

$$\begin{aligned} F(x,y) &= y \cdot f\left(\frac{x}{y}\right) = y\left[\left(\frac{x}{y}\right) - a\left(\frac{x}{y}\right)^2\right] \\ &= x - a\left(\frac{x^2}{y}\right). \end{aligned}$$

Example 20.7

Example 20.7 In a two-factor constant-return-to-scale production process, an econometrician estimates that when the second factor is held constant, the production for the first factor is $f_1(x_1) = x_1^a$ for some $a \in (0, 1)$. Then, the complete production function would be the Cobb-Douglas production function $F(x_1, x_2) = x_1^a x_2^{1-a}$, as we computed in Example 20.5. If units are chosen so that $x_2 = 1$ during the estimation of f_1, then the estimated function is the restriction $f_1(x_1) = F(x_1, 1)$.

Example 20.7

The marginal product of the hidden factor x_2 when $x_2 = 1$ is

$$\frac{\partial F}{\partial x_2}(x_1, 1) = (1-a)x_1^a x_2^{-a}\Big|_{x_2 = 1} = (1 - a)f(x_1)$$

in the specially chosen units of x_2 for which $f_1(x_1) = F(x_1, 1)$.

Homothetic Function

Definition: A function $v : R_+^n \to R$ is called homothetic if it is a monotone transformation of a homogeneous function.

Homothetic Function

Definition: If $X, Y \in R^n$ $X \geq Y$ if $x_i \geq y_i$ for $i = 1, \cdots, n$; $X > Y$ if $x_i > y_i$ for $i = 1, \cdots, n$. A function $u : R^n_+ \to R$ is monotone if for all $X, Y \in R^n$ $X \geq Y \Rightarrow U(X) \geq U(Y)$, is strictly monotone if for all $X, Y \in R^n$ $X > Y \Rightarrow U(X) > U(Y)$,

Homothetic Function

Example 20.13 The two functions at the beginning of this section,

$$v(x,y) = x^3y^3 + xy, \text{ and } w(x,y) = xy + 1,$$

are homothetic functions with $u(x,y) = xy$ and with $g_1(z) = z^3 + z$ and $g_2(z) = z + 1$, respectively. The five examples in Example 20.1 are homothetic functions.

Homothetic Function

Theorem 20.8 Let $u; R^n_+ \to R$ be a strictly monotone function. Then u is homothetic if and only if for all X and Y in R^n_+, $U(X) \geq U(Y) \Leftrightarrow U(\alpha X) \geq U(\alpha Y)$ for all $\alpha > 0$

Homothetic Function

Theorem 20.9 Let u be a C^1 function on R^n_+. If u is homothetic, then the slopes of the tangent planes to the level sets of u are constant along rays from the origin; in other words, for every i, j and for every $X \in R^n$

$$\frac{\frac{\partial U}{\partial x_i}(tX)}{\frac{\partial U}{\partial x_j}(tX)} = \frac{\frac{\partial U}{\partial x_i}(X)}{\frac{\partial U}{\partial x_j}(X)}$$

Homothetic Function

Theorem 20.10 Let u be a C^1 function on R^n_+. If

$$\frac{\frac{\partial U}{\partial x_i}(tX)}{\frac{\partial U}{\partial x_j}(tX)} = \frac{\frac{\partial U}{\partial x_i}(X)}{\frac{\partial U}{\partial x_j}(X)}$$

holds for all $X \in R^n_+$ all $t > 0$, and all i, j, then u is homothetic

Cardinal and Ordinal Utility

- A property of utility function is called **Ordinal** if it depends only on the shape and location of a consumer's indifference sets.
- A property of utility function is called **Cardinal** if it depends on the actual amount of utility that the function assigns to each indifference sets.
- A characteristic of function is called ordinal if every monotonic transformation of a function with this characteristic still has this characteristic. Cardinal properties are not preserved by monotonic transformation.

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Example 20.9

Example 20.9 The function

$$3z+2,\ z^2,\ e^z,\ \ln z$$

are all monotonic transformations of R_{++}, the set of all positive scalers. Consequently, the utility functions

$$3xy+2,\ (xy)^2,\ (xy)^3+xy,\ e^{xy},\ \ln xy=\ln x+\ln y$$

are monotonic transformations of the utility function $u(x,y)=xy$.

Example 20.10

Example 20.10 Consider the class of utility functions on R_+^2 that are monomials$-$polynomials with only one term; for example, the polynomial $u(x, y) = x^2y$. The utility function $v(x, y) = x^2y + 1$ is a monotonic transformation of u. As we discussed above, both u and v have the same indifference curves. However, v is not monomial. So, being monomial is a cardinal property. We should be uncomfortable with any theorem which only holds for monomial utility functions.

Example 20.11

Example 20.11 A utility function $u(x_1, x_2)$ is monotone in x_1 if for each fixed x_2, u is an increasing function x_1. If u id differentiable, we could write this property as $\frac{\partial u}{\partial x_1} > 0$. Intuitively, monotonicity in x_1 means that increasing consumption of commodity one increases utility; in other words, commodity one is a good.

Example 20.11

This property depends only on the shape and location of the level sets of u and on the direction of higher utility. Therefore, it is an ordinal property. Analytically, if $g(z)$ is a monotonic transformation with $g' > 0$, then by the Chain Rule

$$\frac{\partial}{\partial x_1}[g(u(x_1, x_2))] = g'(u(x_1, x_2)) \cdot \frac{\partial u}{\partial x_1}(x_1, x_2) > 0.$$

Example 20.12

Example 20.12 Because of their preferences for ordinal concepts over cardinal concepts, economists would much rather work with the marginal rate of substitution (MRS) than with the marginal utility (MU) of any given utility function, because MU is a cardinal concept. For example, if $v = 2u$

$$\frac{\partial v}{\partial x_1}(x_1^*, x_2^*) = 2\frac{\partial u}{\partial x_1}(x_1^*, x_2^*).$$

Example 20.12

Thus, equivalent utility functions have different marginal utilities at the same bundle. On the other hand, MRS is an ordinal concept. Let v be a general monotonic transformation of $u : v(x,y) = g(u(x,y))$. The MRS for v at

$$\frac{\frac{\partial v}{\partial x}(x^*,y^*)}{\frac{\partial v}{\partial y}(x^*,y^*)} = \frac{\frac{\partial}{\partial x}g(u(x^*,y^*))}{\frac{\partial}{\partial y}g(u(x^*,y^*))} = \frac{\frac{\partial u}{\partial x}(x^*,y^*)}{\frac{\partial u}{\partial y}(x^*,y^*)}$$

the MRS for u at (x^*,y^*).