Mathematical Economics: Lecture 16

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Outline

New Section

Chapter 21: Concave and Quasiconcave Functions

Concave and convex functions

Definition

- A real-valued function f defined on a convex subset U of R^n is **concave** if for all X, Y in U and for all t between 0 and 1, $f(tX + (1-t)Y) \geq tf(X) + (1-t)f(Y)$. Figure 21.2
- A real-valued function f defined on a convex subset U of R^n is **convex** if for all X, Y in U and for all t between 0 and 1, $f(tX + (1-t)Y) \leq tf(X) + (1-t)f(Y)$. Figure 21.3

Concave and convex functions

f is concave if and only if $-f$ is convex

Concave and convex functions

Different with **convex set**: whenever X and Y are points in U, the line segment joining X to Y $l(X, Y) \equiv \{tX + (1-t)Y : 0 \leq t \leq 1\}$ is also in U. Figure 21.1

Concave and convex functions

A function f of n variables is **concave** if and only if any secant line connecting two points on the graph of f lies **below** the graph. A function f of n variables is **convex** if and only if any secant line connecting two points on the graph of f lies **above** the graph.

Concave and convex functions

Theorem 21.1 Let f be a function defined on a convex subset U of R^n. Then, f is concave (convex) if and only if its restriction to very line segment in U is a concave (convex) function of one variable.

Concave and convex functions

Calculus Criteria for Concavity:
Theorem 21.2 Let f be a C^1 function on an interval I in R. Then, f is concave on I if and only if $f(y) - f(x) \leq f'(x)(y - x)$ for all $x, y \in I$. The function f is convex on I if and only if $f(y) - f(x) \geq f'(x)(y - x)$ for all $x, y \in I$.

Concave and convex functions

Theorem 21.3 let f be a C^1 function on a convex subset U of R^n. Then, f is concave on U if and only if for all X, Y in U: $f(Y) - f(X) \leq Df(X)(Y - X)$. Similarly, f is convex on U if and only if for all X, Y in U: $f(Y) - f(X) \geq Df(X)(Y - X)$ for all X, Y in U.

Concave and convex functions

Corollary 21.4 If f is a C^1 concave function on a convex set U and if $X_0 \in U$, then $Df(X_0)(Y - X_0) \leq 0$ implies $f(Y) \leq f(X_0)$. In particular, if $Df(X_0)(Y - X_0) \leq 0$ for all $Y \in U$, then X_0 is a global max of f.

Concave and convex functions

Theorem 21.5 Let f be a C^2 function on an open convex subset U of R^n. Then, f is a concave function on U if and only if the Hessian $D^2 f(X)$ is negative semidefinite for all X in U. The function f is a convex function on U if and only if $D^2 f(X)$ is positive semidefinite for all X in U.

Example 21.1

Example 21.1 Let us apply the test of Theorem 21.3 to show that $f(x_1, x_2) = x_1^2 + x_2^2$ is convex on R^n. The function f is convex if and only if

$$(y_1^2 + y_1^2) - (x_1^2 + x_2^2) \geq (2x_1 \; 2x_2) \begin{pmatrix} y_1 - x_1 \\ y_2 - x_2 \end{pmatrix}$$
$$= 2x_1 y_1 - 2x_1^2 + 2x_2 y_2 - 2x_2^2$$

Example 21.1

if and only if

$$y_1^2 + y_1^2 + x_1^2 + x_2^2 - 2x_1y_1 - 2x_2y_2 \geq 0$$

if and only if

$$(y_1 - x_1)^2 + (y_2 - x_2)^2 \geq 0$$

which is true for all (x_1, x_2) and (y_1, y_2) in R^2.

Example 21.2

Example 21.2 The Hessian of the function $f(x, y) = x^4 + x^2y^2 + y^4 - 3x - 8y$ is

$$D^2 f(x, y) = \begin{pmatrix} 12x^2 + 2y^2 & 4xy \\ 4xy & 12x^2 + 2y^2 \end{pmatrix}$$

For $(x, y) \neq (0, 0)$, the two leading principal minors, $12x^2 + 2y^2$ and $24x^4 + 132x^2y^2 + 24y^4$, are both positive, so f is a convex function on all R^n.

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Example 21.3

Example 21.3 A commonly used simple utility or production function is $F(x, y) = xy$. Its Hessian is

$$D^2F(x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

whose second order principal minor is $\det D^2F(x, y) = -1$. Since this second principal minor is negative, D^2F is indefinite and F is neither concave nor convex.

Example 21.4

Example 21.4 Consider the monotonic transformation of the function F in the previous example by the function $g(z) = z^{1/4} : G(x,y) = x^{1/4}y^{1/4}$, defined only on the positive quadrant R^2_+. The Hessian of G is

$$D^2G(x,y) = \begin{pmatrix} -\frac{3}{16}x^{-7/4}y^{1/4} & \frac{1}{16}x^{3/4}y^{-3/4} \\ \frac{1}{16}x^{-3/4}y^{-3/4} & -\frac{3}{16}x^{1/4}y^{-7/4} \end{pmatrix}$$

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Example 21.4

For $x > 0, y > 0$, the first order leading principal minor is negative and the second order leading principal minor, $x^{-3/2}y^{-3/2}/128$, is positive. Therefore, $D^2G(x, y)$ is negative definite on R^2_+ and G is a concave function on R^2_+.

Example 21.5

Example 21.5 Now, consider the general Cobb-Douglas function on R^2_+ : $U(x, y) = x^a y^b$. Its Hessian is

$$D^2U(x, y) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix}$$

Example 21.5

Example 21.5 Now, consider the general Cobb-Douglas function on $R^2_+ : U(x, y) = x^a y^b$. Its Hessian is

$$D^2U(x, y) = \begin{pmatrix} a(a-1)x^{a-2}y^b & abx^{a-1}y^{b-1} \\ abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix}$$

Example 21.5

whose determinant is

$$det D^2 U(x, y) = ab(1 - a - b)x^{2a-2}y^{2b-2}.$$

In order for U to be concave on R^2_+, we need $a(a-1) < 0$ and $ab(1-a-b) > 0$; that is, we need $0 < a < 1, 0 < b < 1$, and $a + b \leq 1$. In summary, a Cobb-Douglas production function on R^2_+ is concave if and only if it exhibits constant or decreasing returns to scale.

Properties of Concave functions

Theorem 21.6 Let f be a concave (convex) function on an open convex subset U of R^n. If x_0 is a critical point of f, that is, $Df(x_0) = 0$, then $x_0 \in U$ is a global maximizer (minimizer) of f on U.

Properties of Concave functions

Theorem 21.7 Let f be a C^1 function defined on a convex subset U of R^n. If f is a concave function and if x_0 is a point in U which satisfies $Df(x_0)(y - x_0) \leq 0$ for all $y \in U$, then x_0 is a global maximizer of f on U.

Example 21.6

Example 21.6 If f is a C^1 increasing, concave function of one variable on the interval [a,b], then $f'(b)(x-b) \leq 0$ for all $x \in [a,b]$. By Theorem 21.7, b is the global maximizer of f on [a,b].

Example 21.7

Example 21.7 Consider the concave function $U(x, y) = x^{1/4}y^{1/4}$ on the (concave) triangle

$$B = \{(x, y) : x \geq 0, y \geq 0, x + y \leq 2\}.$$

By symmetry, we would expect that $(x_0, y_0) = (1, 1)$ is the maximizer of U on B. To prove this, use Theorem 21.7. Let (x,y) be an arbitrary point in B.

Example 21.7

$$\begin{aligned}
\frac{\partial U}{\partial x}(1,1)(x-1)+\frac{\partial U}{\partial y}(1,1)(y-1) \\
=\frac{1}{4}(x-1)+\frac{1}{4}(y-1) \\
=\frac{1}{4}(x+y-2) \\
\leq 0
\end{aligned}$$

since $x+y-2 \leq 0$ for (x,y) in the constraint set B. By Theorem 21.7, (1,1) is the global maximizer of U on B.

Example 21.8

Example 21.8 Consider the problem of maximizing profit for a firm whose production function is $y = g(x)$, where y denotes output and x denotes the input bundle. If p denotes the price of the output and w_i is the cost per unit of input i, then the firm's profit function is

$$\Pi(x) = pg(x) - (w_1 x_1 + \cdots + w_n x_n)$$

Example 21.8

As can easily be checked, Π will be a concave function provided that the production function is a concave function. In this case, the first order condition

$$p\frac{\partial g}{\partial x_i} = w_i \text{ for } i = 1, 2, \ldots, n,$$

which says marginal revenue product equals the factor price for each point, is both necessary and sufficient for an interior profit maximizer.

Example 21.8

If one wants to study the effect of changes in w_i or p on the optimal input bundle, one would apply the comparative statics analysis to system. Since profit is concave for all p and w, the solution to system will automatically be the optimal input for all choices of p and w.

Properties of Concave functions

Theorem 21.8 Let $f_1. \cdots, f_k$ be concave (convex) functions. each defined on the same convex subset U of R^n. Let $a_1, \cdots, a_k$ be positive numbers. Then, $a_1 f_1 + \cdots + a_k f_k$ is a concave (convex) function on U.

Properties of Concave functions

Theorem 21.9 let f be a function defined on a convex set U in R^n. If f is concave, then for every x_0 in U, the set $C^+_{x_0} \equiv \{x \in U : f(x) \geq f(x_0)\}$ is a convex set. If f is convex, then for every x_0 in U, the set $C^-_{x_0} \equiv \{x \in U : f(x) \leq f(x_0)\}$ is a convex set.

Quasiconcave and Quasiconvex

- Definition: a function f defined on a convex subset U of R^n is quasiconcave if for every real number a, $C_a^+ \equiv \{x \in U : f(x) \geq a\}$ is a convex set. Similarly, f is quasiconvex if for every real number a, $C_a^- \equiv \{x \in U : f(x) \leq f(x_0)\}$ is a convex set.
- Figure 21.9

Quasiconcave and Quasiconvex

Theorem 21.12 Let f be a function defined on a convex set U in R^n. Then, the following statements are equivalent to each other:

(a) f is a quasiconcave function on U

(b) For all $X, Y \in U$ and all $t \in [0, 1]$ $f(X) \geq f(Y)$ implies $f(tX + (1 - t)Y) \geq f(Y)$

(c) For all $X, Y \in U$ and all $t \in [0, 1]$ $f(tX + (1 - t)Y) \geq \min\{f(Y), f(X)\}$.

Example 21.9

Example 21.9 Consider the Leontief or fixed-coefficient production function $Q(x, y) = \min\{ax, by\}$ with $a, b > 0$. The level sets of Q are drawn in Figure 21.7. Certainly, the region above and to the right of any of this function's L-shaped level sets is a convex set. Q is quasiconcave.

Extra Theorem

Theorem 21.* Any monotonic transformation of a concave function is quasiconcave.

Quasiconcave and Quasiconvex

Theorem 21.13 Every Cobb-Douglas function $F(x, y) = Ax^a y^b$ with A, a and b all positive is quasiconcave.

Example 21.10

Example 21.10 Consider the constant elasticity of substitution (CES) production function

$$Q(x,y) = (a_1 x_1^r + a_2 x_2^r)^{1/r}, \text{ where } 0 < r < 1.$$

By Theorem 21.8 and Exercise 21.4, $(a_1 x_1^r + a_2 x_2^r)$ is concave. Since $g(z) = z^{1/r}$ is a monotonic transformation, Q is a monotonic transformation of a concave function and therefore is quasiconcave.

Example 21.11

Example 21.11 Let $y = f(x)$ be any increasing function on R^1, as in Figure 21.8. For any x^*, $\{x : f(x) \geq f(x^*)\}$ is just the interval $[x^*, \infty)$, a convex subset of R^1. So, f is quasiconcave. On the other hand, $\{x : f(x) \leq f(x^*)\}$ is the concave set $(\infty, x^*]$. Therefore, an increasing function on R^1 is both quasiconcave and quasiconvex. The same argument applies to decreasing function.

Example 21.12

Example 21.12 Any function on R^1 which rises monotonically until it reaches a global maximum and then monotonically falls, such as $y = -x^2$ or the bell-shaped probability density function $y = ke^{-x^2}$, is a quasiconcave function, as Figure 21.9 indicates. For any x_1 as in Figure 21.9, there is a x_2 such that $f(x_1) = f(x_2)$. Then, $\{x : f(x) \geq f(x_1)\}$ is the convex interval $[x_1, x_2]$.

Quasiconcave and quasiconvex

Calculus Criteria: **Theorem 21.14** Suppose that F is a C^1 function on an open convex subset U of R^n. Then, F is quasiconcave on U if and only if $F(y) \geq F(x)$ implies that $DF(x)(y - x) \geq 0$; F is quasiconvex on U if and only if $F(y) \leq F(x)$ implies that $DF(x)(y - x) \leq 0$;

Quasiconcave and quasiconvex

Theorem 21.15 Suppose that F is a real-valued positive function defined on a convex cone C in R^n. If F is homogeneous of degree one and quasiconcave on C, it is concave on C.

Pseudoconcave and Pseudoconvex

Definition : Let U be an open convex subset of R^n. A C^1 function $F : U \to R$ is pseudoconcave at $x^* \in U$ if $DF(x^*)(y - x^*) \leq 0$ implies $F(y) \leq F(x^*)$ for all $y \in U$. The function F is pseudoconcave on U if (15) holds for all $x^* \in U$. To define a pseudoconvex function on U, one simply reverses **all** inequalities.

Pseudoconcave and Pseudoconvex

Theorem 21.16 Let U be a convex subset of R^n, and let $F : U \to R$ be a C^1 pseudoconcave function. If $x^* \in U$ has the property $DF(x^*)(y - x^*) \leq 0$ for all $y \in U$, for example, $DF(x^*) = 0$, then x^* is a global max of F on U. An analogous result holds for pseudoconvex functions.

Pseudoconcave and Pseudoconvex

Theorem 21.17 Let U be a convex subset of R^n. Let $F : U \to R$ be a C^1 function. Then, (a) if F is pseudoconcave on U, F is quasiconcave on U, and (b) if U is open and if $\nabla F(x) \neq 0$ for all $x \in U$, then F is pseudoconcave on U if and only if F is quasiconcave on U.

Pseudoconcave and Pseudoconvex

Theorem 21.18 Let U be an open convex subset of R^n. Let $F : U \to R$ be a C^1 function on U. Then, F is pseudoconcave on U if and only if for each x^* in U, x^* is the solution to the constrained maximization problem $\max F(x)$ s.t $C_{x^*} \equiv \{y \in U : DF(x^*)(y - x^*) \leq 0\}$.

Pseudoconcave and Pseudoconvex

Theorem 21.19 Let F be a C^2 function on an open convex subset W in R^n. Consider the bordered Hessian H (a) If the largest (n-1) leading principal minors of H alternate in sign, for all $x \in W$, with the smallest of these positive, then F is pseudoconcave, and therefore quasiconcave, on W. (b) If these largest (n-1) leading principal minors are all negative for all $x \in W$, then F is pseudoconvex, and therefore quasiconvex, on W.

Pseudoconcave and Pseudoconvex

Theorem 21.20 Let F be a C^2 function on a convex set W in R^2. Suppose that F is monotone in that $F'_x > 0$ and $F'_y > 0$ on W. If the determinant (18) is positive for all $(x, y) \in W$, then F is quasiconcave on W. If the determinant (19) is negative for all $(x, y) \in W$, then F is quasiconvex on W. Conversely, if F is quasiconcave on W, then the determinant (19) is positive; if F is quasiconvex on W, then the determinant (19) is ≤ 0 for all $(x, y) \in W$.

Example 21.13

Example 21.13 Theorem 21.13 implies that the Cobb-Douglas function $U(x, y) = x^a y^b$ is quasiconcave on R^2_+ for $a, b > 0$ since it is a monotone transformation of a concave function. Let's use Theorem 21.20 to prove the quasiconcavity of U. The bordered Hessian is

$$\begin{pmatrix} 0 & ax^{a-1}y^b & bx^a y^{b-1} \\ ax^{a-1}y^b & a(a-1)x^{a-1}y^b & abx^{a-1}y^{b-1} \\ bx^a y^{b-1} & abx^{a-1}y^{b-1} & b(b-1)x^a y^{b-2} \end{pmatrix}$$

Example 21.13

whose determinant is

$$(ab + ab^2 + a^2 b)x^{3a-2}y^{3b-2},$$

which is always positive for $x > 0, y > 0, a > 0, b > 0$. By Theorem 21.20, U is pseudoconcave, and therefore quasiconcave.

Concave Programming

Unconstrained Problems : **Theorem 21.21** Let U be a convex subset of R^n. Let $f : U \to R$ be a C^1 concave (convex) function on U. Then, x^* is a global max of f on U if and only if $Df(x^*)(x - x^*) \leq 0$ for all $x \in U$. In particular, if U is open, or if x^* is an interior point of U, then x^* is a global max(min) of f on U if and only if $Df(x^*) = 0$

Concave Programming

Constrained Problem : **Theorem 21.22** Let U be a convex open subset of R^n. Let $f : U \to R$ be a C^1 pseudoconcave function on U. Let $g_1, \cdots, g_k : U \to R$ be C^1 quasiconvex functions. If (x^*, λ^*) satisfy the Lagrangian conditions, x^* is a global max of f on the constraint set.

Concave Programming

Theorem 21.23 let $f, g_1, \cdots, g_k$ be as in the hypothesis of Theorem 21.22. (a) For any fixed $b = (b_1, \cdots, b_k) \in R^k$, let $Z(b)$ denote the set of $x \in C_b$ that are global maximizers of f on C_b. Then, $Z(b)$ is a convex set. (b) For any $b \in R^k$, let $V(b)$ denote the maximal value of the objective function f in problem (20). If f is concave and the g_i are convex, then $b \to V(b)$ is a concave function of b.

Concave Programming

Saddle Point: Definition: Let U be a convex subset of R^n. Consider the Lagrangian function (21) for the programming problem (20), as a function of x and λ. Then, (x^*, λ) is saddle point of L if $L(x, \lambda^*) \leq L(x^*, \lambda^*) \leq L(x^*, \lambda)$ for all $\lambda \geq 0$ and all $x \in U$. Usually, $U = R^n$ or $U = R^n_+$, the positive orthant of R^n. In the latter case, we say that (x^*, λ^*) is a nonnegative saddle point of L

Concave Programming

Theorem 21.24 If (x^*, λ^*) is a (nonnegative) saddle point for L in Problem (20), then x^* maximizes f on $C_b(C_b \cap R^n_+)$.

Concave Programming

Theorem 21.25 Suppose that $U = R^n_+$ or that U is an open convex subset of R^n. Suppose that f is a C^1 concave function and that $g_1, \cdots, g_k$ are C^1 convex functions on U. Suppose that x^* maximizes f on the constraint set C_b as defined in (20). Suppose further that one of the constraint qualifications in Theorem 19.12 holds. Then, there exists $\lambda^* > 0$ such that (X^*, λ^*) is a saddle point of the Lagrangian (21)