Mathematical Economics: Lecture 17

Yu Ren

WISE, Xiamen University

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Outline

New Section

Chapter 23: Eigenvalues and Dynamics

Definitions

- Eigenvalue: Let A be a square matrix. An eigenvalue of A is a number r which makes $det(A - rI) = 0$

Example 23.1

Example 23.1 Consider the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

Subtracting 2 from each diagonal entry transforms A into the singular matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Therefore, 2 is an eigenvalue of A.

Example 23.2

Example 23.2 Let's look for the eigenvalues of the diagonal matrix $B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ Subtracting a 2 from each of the diagonal entries yields the singular matrix $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ subtracting a 3 from each of the diagonal entries yields the singular matrix $\begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, 2 and 3 are eigenvalues od the $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

Theorem

- **Theorem 23.1** The diagonal entries of a diagonal matrix D are eigenvalues of D.
- **Theorem 23.2** A square matrix A is singular if and only if 0 is an eigenvalue of A.

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Example 23.3

Example 23.3 Consider the matrix $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Since the first row is the negative of the second, B is a singular matrix and, therefore, 0 is an eigenvalue of B. We can use the observation in Example 23.1 to find a second eigenvalue, because subtracting 2 from each diagonal entry of B yields the singular matrix $\begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix}$. We conclude that 0 and 2 are eigenvalues of B.

Example 23.4

Example 23.4 A matrix M whose entries are nonnegative and whose columns (or rows) each add to 1, such as

$$B = \begin{pmatrix} 1/4 & 2/3 \\ 3/4 & 1/3 \end{pmatrix}$$

ia called a Markov matrix. As we will see in Section 23.6, Markov matrices play a major role in the dynamics of economic systems.

Example 23.4

If we subtract a 1 from each diagonal entry of the Markov matrix, then each column of the transformed matrix:

$$M - 1I = \begin{pmatrix} -3/4 & 2/3 \\ 3/4 & -2/3 \end{pmatrix}$$

adds up to 0. But if the columns of a square matrix add up to $(0, \ldots, 0)$, the rows are linearly dependent and the matrix must be singular. It follows that $r = 1$ is an eigenvalue of matrix B. The same argument shows that $r = 1$ is an eigenvalue of every Markov matrix.

Theorem

Theorem 23.3 Let A be an $n \times n$ matrix and let r be a scalar. Then, the following statements are equivalent:

- (a) Subtracting r from each diagonal entry of A transforms A into a singular matrix.
- (b) $A - rI$ is a singular matrix
- (c) det$(A - rI)$=0
- (d) $(A - rI)V = 0$ for some nonzero vector V.
- (e) $AV = rV$ for some nonzero vector V.

Theorem

- V is called eigenvector corresponding to eigenvalue r

Example 23.5

Example 23.5 Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} -1 & 3 \\ +2 & 0 \end{pmatrix}$$

Its characteristic polynomial is

$$det(A - rI) = det \begin{pmatrix} -1 - r & 3 \\ 2 & 0 - r \end{pmatrix} = (r+3)(r-2).$$

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Example 23.5

The eigenvalues of A are the roots of the characteristic polynomial: -3 and 2. To find the corresponding eigenvectors, use statement d of Theorem 23.3. First, subtract eigenvalue -3 from the diagonal entries of A and solve

$$(A-(-3)I)v = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for v_1 and v_2.

Example 23.5

For a 2×2 matrix, these equations are easily solved just by looking carefully at the equation. For example, we can take $v_1 = 3$ and $v_2 = -2$ and conclude that one eigenvector is $\binom{3}{-2}$. There are other eigenvalues, such as

$$\begin{pmatrix} 1 \\ -2/3 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 1 \end{pmatrix}$$

In general, one chooses the *"simplest"* of the nonzero candidates. The set of all solutions of linear equation$-$including $v = 0$$-$is called the eigenspace of A with respect to -3.

Example 23.5

To find the eigenvector for eigenvalue $r = 2$, subtract 2 from the diagonal entries of A

$$(A - 2I)v = \begin{pmatrix} -3 & 3 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The simplest solution is $\binom{1}{1}$;but any multiple of $\binom{1}{1}$ is also an eigenvector for 2. The eigenspace for eigenvalue 2 is the diagonal line in R^2.

Example 23.6

Example 23.6 Example 23.6 Let's compute the eigenvalues and eigenvectors of the 3X3 matrix

$$B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 5 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

Its characteristic equation is

$$det \begin{pmatrix} 1-r & 0 & 2 \\ 0 & 5-r & 0 \\ 3 & 0 & 2-r \end{pmatrix} = (5-r)(r-4)(r+1)$$

Example 23.6

Therefore, the eigenvalues of B are $r = 5, 4, -1$.
To compute an eigenvector corresponding to $r = 5$, we compute the nullspace of $(B - 5I)$; that is, we solve the system

$$(B - 5I)v = \begin{pmatrix} -4 & 0 & 2 \\ 0 & 0 & 0 \\ 3 & 0 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -4v_1 + 2v_2 \\ 0 \\ 3v_1 - 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

whose solution is $v_1 = v_3 = 0$, $v_2 =$ anything.

Example 23.6

So, we'll take $v_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ as an eigenvector for $r = 5$.

Example 23.6

To find an eigenvector for $r = 4$, solve

$$(B - 4I)v = \begin{pmatrix} -3 & 0 & 2 \\ 0 & 1 & 0 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$= \begin{pmatrix} -4v_1 + 2v_2 \\ 0 \\ 3v_1 - 3v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Example 23.6

This system reduces to the two equations

$$\begin{aligned} -3v_1 + 2v_3 &= 0 \\ v_2 &= 0 \end{aligned}$$

A simple eigenvector for $r = 4$ is $\begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix}$. This same method yields the eigenvector $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ for eigenvalue $r = -1$.

Solving Linear Difference Equations

$$\begin{aligned} y_{n+1} &= a y_n \\ y_1 &= a y_0 \\ y_n &= a^n y_0 \end{aligned}$$

Solving Linear Difference Equations

$$x_{n+1} = ax_n + by_n$$
$$y_{n+1} = cx_n + dy_n$$

Example 23.7

Example 23.7 Consider the coupled system of difference equations generated by a Leslie model with $b_1 = 1, b_2 = 4$ and $d_1 = 0.5$

$$
\begin{aligned}
x_{n+1} &= 1x_n + 4y_n \\
y_{n+1} &= 0.5x_n + 0y_n
\end{aligned}
$$

Example 23.7

The right change of coordinates for solving this system is

$$\begin{aligned} X &= \frac{1}{6}x + \frac{1}{3}y \\ Y &= -\frac{1}{6}x + \frac{2}{3}y \end{aligned}$$

whose inverse transformation is

$$\begin{aligned} x &= 4X - 2Y \\ y &= X + Y \end{aligned}$$

Example 23.7

In matrix form,

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix}$$

These two matrices are necessarily inverse to each other.

Example 23.7

Use these transformations

$$X_{n+1} = \frac{1}{6}x_{n+1} + \frac{1}{3}y_{n+1} = \frac{1}{6}(x_n + 4y_n) + \frac{1}{3}\left(\frac{1}{2}x_n\right)$$

$$Y_{n+1} = -\frac{1}{6}x_{n+1} + \frac{2}{3}y_{n+1} = -\frac{1}{6}(x_n + 4y_n) + \frac{2}{3}\left(\frac{1}{2}x_n\right.$$

using the change of coordinates and then difference equation.

Example 23.7

Simplifying,

$$\begin{aligned}
X_{n+1} &= \frac{1}{3}x_n + \frac{2}{3}y_n = \frac{1}{3}(4X_n - 2Y_n) + \frac{2}{3}(X_n + Y_n) \\
&= 2X_n \\
Y_{n+1} &= \frac{1}{6}x_n - \frac{2}{3}y_n = \frac{1}{6}(4X_n - 2Y_n) - \frac{2}{3}(X_n + Y_n) \\
&= -Y_n
\end{aligned}$$

Example 23.7

then

$$\begin{aligned} X_{n+1} &= 2X_n \\ Y_{n+1} &= -Y_n \end{aligned}$$

is completely uncoupled and, so, can be easily solved as two one-dimensional equations:

$$\begin{aligned} X_n &= 2^n c_1 \\ Y_n &= (-1)^n c_2 \end{aligned}$$

Example 23.7

Finally,

$$\begin{aligned} x_n &= 4X_n - 2Y_n = 4 \cdot 2^n c_1 - 2 \cdot (-1)^n c_2 \\ y_n &= X_n + Y_n = 2^n c_1 + (-1)^n c_2 \end{aligned}$$

which can be written in vector notation as

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} 4 \cdot 2^n c_1 - 2 \cdot (-1)^n c_2 \\ 2^n c_1 + (-1)^n c_2 \end{pmatrix}$$
$$= c_1 2^n \begin{pmatrix} 4 \\ 1 \end{pmatrix} + c_2 (-1)^n \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

Example 23.7

The constants c_1 and c_2 are determined by the initial amounts x_0 and y_0.

$$\begin{aligned} x_0 &= 4c_1 - 2c_2 \\ y_0 &= c_1 + c_2 \end{aligned}$$

This system of equations can be solved in the usual way:

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{3} \\ -\frac{1}{6} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Solving Linear Difference Equations

- $z_{t+1} = Az_n$
- $z = PZ$

Solving Linear Difference Equations

$$
\begin{aligned}
Z_{n+1} &= P^{-1} z_{n+1} \\
&= P^{-1}(A z_n) \\
&= (P^{-1} A) z_n \\
&= (P^{-1} A) P Z_n \\
&= (P^{-1} A P) Z_n
\end{aligned}
$$

Solving Linear Difference Equations

Solution: **Theorem 23.4** Let A be a $k \times k$ matrix. Let $r_1, r_2, \cdots, r_k$ be eigenvalues of A, and $V_1, V_2, \cdots, V_k$ the corresponding eigenvalues. Form the matrix $P = [V_1, V_2, \cdots, V_k]$ whose columns are these k eigenvectors. If P is invertible, then $P^{-1}AP = diag(r_k)$. Conversely, if $P^{-1}AP$ is a diagonal matrix D, the columns of P must be eigenvectors of A and the diagonal entries of D must be eigenvalues of A.

Solving Linear Difference Equations

Revisit example 23.7 specify r, P, A

Solving Linear Difference Equations

Theorem 23.5 Let $r_1, \cdots, r_h$ be h distinct eigenvalues of the $k \times k$ matrix A. let $V_1, \cdots, V_h$ be corresponding eigenvectors. Then $V_1, \cdots, V_h$ are linearly independent, that is, no one of them can be written as a linear combination of the others.

Solving Linear Difference Equations

- **Theorem 23.6** Let A be a $k \times k$ matrix with k distinct real eigenvalues $r_1, \cdots, r_k$ and corresponding eigenvectors $V_1, \cdots, V_k$. The general solution of the system of difference equations $z_{n+1} = Az_n$ is $z_n = c_1 r_1^n v_1 + c_2 r_2^n v_2 + \cdots + c_k r_k^n v_k$
- Page 594 and **Theorem 23.7**: Alternative Approach

Solving Linear Difference Equations

- **Theorem 23.6** Let A be a $k \times k$ matrix with k distinct real eigenvalues $r_1, \cdots, r_k$ and corresponding eigenvectors $V_1, \cdots, V_k$. The general solution of the system of difference equations $z_{n+1} = Az_n$ is $z_n = c_1 r_1^n v_1 + c_2 r_2^n v_2 + \cdots + c_k r_k^n v_k$
- Page 594 and **Theorem 23.7**: Alternative Approach

Example 23.8

Example 23.8 We computed in Example 23.5 that the eigenvalues of $C = \begin{pmatrix} -1 & 3 \\ +2 & 0 \end{pmatrix}$ are $-3, 2$ with corresponding eigenvectors $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Example 23.8

By Theorem 23.4

$$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

Example 23.8

By Theorem 23.7,

$$\begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix}^n =$$

$$\begin{pmatrix} 3 & 1 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} (-3)^n & 0 \\ 0 & 2^n \end{pmatrix} \begin{pmatrix} 0.2 & -0.2 \\ 0.4 & 0.6 \end{pmatrix}.$$

Stability

- Stability of equilibria
- **Theorem 23.8** If the $k \times k$ matrix A has k distinct real eigenvalues, then every solution of the general system of linear difference equations (23) tends to 0 if and only if all the eigenvalues of A have absolute value less than 1.

Properties of Eigenvalues

- $p(r)$ has k distinct, real roots
- $p(r)$ has some repeated roots
- $p(r)$ has some complex roots

Example 23.10

Example 23.10

$$(a) \quad \text{For matrix } \begin{pmatrix} -4 & 2 \\ -1 & -1 \end{pmatrix}$$

the characteristic polynomial is $p_1(r) = (-4 - r)(-1 - r) = r^2 + 5r + 6$, whose roots are the distinct, real number $r = -3, -2$.

Example 23.10

(b)

$$\begin{pmatrix} 4 & 0 & -2 \\ 0 & 3 & 0 \\ 3 & 0 & -1 \end{pmatrix}$$

is
$p_2(r) = (4-r)(3-r)(-1-r) = (3-r)(r^2-3r+2)$,
whose roots are the distinct, real number
$r = 1, 2, 3$.

Example 23.10

(c)

$$\begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

The characteristic polynomial of the matrix is
$p_3(r) = (4-r)(2-r) + 1 = (r-3)^2$,
whose roots are $r = 3, 3$, 3 is a root of $p_3(r)$ of multiplicity two.

Example 23.10

(d)

$$\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

The characteristic polynomial of the matrix is
$p_4(r) = (3 - r)^2$
Once again, 3 is an eigenvalue of multiplicity two.

Example 23.10

(e)

$$\begin{pmatrix} 0 & 2 \\ -1 & 2 \end{pmatrix}$$

The characteristic polynomial of the matrix is
$p_5(r) = -r(2-r) + 2 = r^2 - 2r + 2,$
Its roots are the complex numbers
$r = 1+i, 1-i.$

Repeated Eigenvalues

Theorem 23.9 Let A be a $k \times k$ matrix with eigenvalues $r_1, r_2, \cdots, r_k$. Then
(a) $r_1 + r_2 + \cdots + r_k = tr(A)$
(b) $r_1 \cdot r_2 \cdots r_k = det(A)$

Example 23.11

Example 23.11 To find the eigenvalues of the 4X4 matrix

$$B = \begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 4 \end{pmatrix},$$

subtract 3 from each of the diagonal entries of B.

Example 23.11

The result is the singular matrix

$$B - 3I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

Example 23.11

Vector $v = (v_1, v_2, v_3, v_4)$ is an eigenvector of eigenvalue 3 if and only if $(B - 3I)v = 0$ if and only if $v_1 = v_2 = v_3 = v_4 = 0$. The vectors

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \; v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

are three linearly independent eigenvectors of eigenvalue 3.

Example 23.11

Therefore, 3 ia an eigenvalue of B of multiplicity at least 3. Use the fact that the sum of the eigenvalues of B is 16, the trace of B, to conclude that the fourth eigenvalue of B is $16 - (3 + 3 + 3) = 7$.

Repeated Eigenvalues

- Nondiagonalizable matrix (page 601)
- **Theorem 23.10**: Let A be a 2×2 matrix with two equal eigenvalues. Then, A is diagonizable if and only if A is already diagonal.

Repeated Eigenvalues

- almost diagonal matrix:

$$\begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

- Is this form simple enough that we can always easily solve the transformed system?
- Is this “almost diagonal” form achieve as $P^{-1}AP$ for any defective matrix A?

Repeated Eigenvalues

$$P^{-1}AP = \begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

$$A[v_1\ v_2] = [v_1\ v_2]\begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

Repeated Eigenvalues

$$
\begin{aligned}
[Av_1\ Av_2] &= [r^* v_1\ v_1 + r^* v_2] \\
(A - r^* I)v_1 &= 0 \\
(A - r^* I)v_2 &= v_1
\end{aligned}
$$

Example 23.13

Example 23.13 Consider the nondiagonalizable matrix

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}$$

of Example 23.10c, whose eigenvalues are $r = 3, 3$. As we computed earlier, it has one independent eigenvector $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Example 23.13

Its generalized eigenvector will be a solution v_2 of

$$(A-3I)v_2 = v_1, \text{ or } \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} v_{21} \\ v_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Take $v_{21} = 1, v_{22} = 0$, for example. Then form

$$P = [v_1 \ v_2] = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

Example 23.13

and check that

$$P^{-1}AP = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$$

Repeated Eigenvalues

Theorem 23.11 let A be a 2×2 matrix with equal eigenvalue $r = r^*, r^*$. Then, (a) either A has two independent eigenvectors corresponding to r^*, in which case A is diagonal matrix $r^* I$ or (b) A has only one independent eigenvector, say V_1. In this case, there is a generalized eigenvector V_2 such that $(A - r^* I)V_2 = V_1$. If $P \equiv [V_1 V_2]$ then

$$P^{-1}AP = \begin{pmatrix} r^* & 1 \\ 0 & r^* \end{pmatrix}$$

Example 23.14

Example 23.14 The characteristic polynomial of the matrix

$$A = \begin{pmatrix} 4 & 2 & -4 \\ 1 & 4 & -3 \\ 1 & 1 & 0 \end{pmatrix}$$

is $p(r) = (r-3)^2(2-r)$, its eigenvalues are $r = 3, 3, 2$.

Example 23.14

For eigenvalues $r = 2$, the solution space of

$$(A - 2I)v = \begin{pmatrix} 2 & 2 & -4 \\ 1 & 2 & -3 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is the one-dimensional space spanned by

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Example 23.14

For eigenvalue $r = 3$, the solution space of

$$(A - 3I)v = \begin{pmatrix} 1 & 2 & -4 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

is the one-dimensional space spanned by

$$v_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}.$$

Example 23.14

There is only one independent eigenvector corresponding to the eigenvalue of multiplicity two. We need one more vector v_3 independent of v_1, v_2 to form the change of coordinate matrix $P = [v_1, v_2, v_3]$. Take v_3 to be a generalized eigenvector for the eigenvalue $r = 3$–a solution to the system

$$(A - 3I)v_3 = v_2, \text{ or } \begin{pmatrix} 1 & 2 & -4 \\ 1 & 1 & -3 \\ 1 & 1 & -3 \end{pmatrix} \begin{pmatrix} v_{31} \\ v_{32} \\ v_{33} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

Example 23.14

By inspection, we can take $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

Example 23.14

Let

$$P = [v_1 v_2 v_3] = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Then,

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Repeated Eigenvalues

- page 606
- **Theorem 23.12** Suppose that A is a 2×2 matrix with multiple eigenvalue r and only one independent eigenvector V_1. Let V_2 be a generalized eigenvector corresponding to V_1 and r. Then general solution of the system of difference equation $z_{n+1} = Az_n$ is $z_n = (c_0 r^n + nc_1 r^{n-1})V_1 + c_1 r^n V_2$

Example 23.15

Example 23.15 The linear system of difference equations corresponding to the nondiagonalizable matrix in Example 23.13 is

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Example 23.15

By Theorem 23.12, its general solution is

$$\begin{aligned}\begin{pmatrix} x_n \\ y_n \end{pmatrix} &= (c_0 3^n + c_1 n 3^{n-1}) \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_1 3^n \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} c_0 3^n + c_1(n3^{n-1} + 3^n) \\ -c_0 - c_1 n 3^{n-1} \end{pmatrix}.\end{aligned}$$

Complex eigenvalues

Example 23.16 The eigenvalues of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the general 2X2 matrix, are the roots of its characteristic equation

$$p(r) = r^2 - (a + d)r + (ad - bc),$$

Example 23.16

namely,

$$\begin{aligned} r &= \tfrac{1}{2}(a+d) \pm \tfrac{1}{2}\sqrt{(a+d)^2 - 4(ad-bc)} \\ &= \tfrac{1}{2}(a+d) \pm \tfrac{1}{2}\sqrt{(a-d)^2 + 4bc} \end{aligned}$$

If $(a-d)^2 + 4bc < 0$, then the roots are the complex numbers

$$\begin{aligned} r_1 &= \tfrac{1}{2}(a+d) \pm i\tfrac{1}{2}\sqrt{|(a+d)^2 + 4bc|} \\ r_2 &= \bar{r}_1 = \tfrac{1}{2}(a+d) \pm -i\tfrac{1}{2}\sqrt{|(a+d)^2 + 4bc|} \end{aligned}$$

Complex eigenvalues

1. (37) (38)
2. **Theorem 23.13** Let A be a $k \times k$ matrix with real entries. If $r = \alpha + i\beta$ is an eigenvalue of A, so is its conjugate $\bar{r} = \alpha - i\beta$. If $U + iV$ is an eigenvector for $\alpha + i\beta$, then $u - iv$ is an eigenvector for $\alpha - i\beta$. If k is odd, A must have at least one real eigenvalue.

Example 23.17

Example 23.17 For matrix $A = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}$, the characteristic polynomial is

$$p(r) = r^2 - 2r + 10,$$

whose roots are $r = 1 + 3i, 1 - 3i$. An eigenvector for $r = 1 + 3i$ is a solution w of

$$(A-(1+3i)I)w = \begin{pmatrix} -3i & 1 \\ -9 & -3i \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Example 23.17

This matrix may not look singular, but its determinant is zero and its second row is $-3i$ times its first row. Using the first row of this matrix, we conclude that an eigenvector is a solution w of the equatio

$$-3iw_1 + w_2 = 0;$$

for example, $w = \begin{pmatrix} 1 \\ -3i \end{pmatrix}$, which we write as $\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Example 23.17

By Theorem 23.13, an eigenvector for eigenvalue $1 - 3i$ is

$$\bar{w} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - i \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ -3i \end{pmatrix}$$

Example 23.17

From the change of coordinate matrix P whose columns are these two eigenvectors:

$$P = \begin{pmatrix} 1 & 1 \\ 3i & -3i \end{pmatrix}$$

Then, applying Theorem 8.5.4

$$P^{-1} = -\frac{1}{6i}\begin{pmatrix} -3i & -1 \\ -3i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{6}i \\ \frac{1}{2} & \frac{1}{6}i \end{pmatrix},$$

Example 23.17

and we calculate that

$$P^{-1}AP = \begin{pmatrix} 1+3i & 0 \\ 0 & 1-3i \end{pmatrix}$$

just as if we had been working with real eigenvalues and real eigenvectors.

Complex eigenvalues

- (40) (41) Figure 23.2
- **Theorem 23.14** Let A be a real 2×2 matrix with complex eigenvalues $\alpha^* \pm i\beta^*$ and corresponding complex eigenvectors $u^* \pm iv^*$. Write the eigenvalues $\alpha^* \pm i\beta^*$ in polar coordinates as $r^*(\cos\theta^* + i\sin\theta^*)$. Then, the general solution of the difference equation $z_{n+1} = Az_n$ is

$$\begin{aligned} z_n &= {r^*}^n[(C_1 \cos n\theta^* - C_2 \sin n\theta^*)u^* \\ &- (C_2 \cos n\theta^* + C_1 \sin n\theta^*)v^*] \end{aligned}$$

Example 23.18

Example 23.18 In example 23.17, we computed that the eigenvalues of $A = \begin{pmatrix} 1 & 1 \\ -9 & 1 \end{pmatrix}$ are $1 \pm 3i$ with corresponding eigenvectors $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 0 \\ 3 \end{pmatrix}$.

Example 23.18

In polar coordinates,

$$1+3i = \sqrt{10}\left(\frac{1}{\sqrt{10}} + i\frac{3}{\sqrt{10}}\right) = \sqrt{10}(\cos\theta^* + \sin\theta^*),$$

where $\theta^* = \arccos(1/\sqrt{10}) \approx 71.565^o$ or 1.249 radians.

Example 23.18

The general solution of

$$\begin{aligned} x_{n+1} &= x_n + y_n \\ y_{n+1} &= -9x_n + y_n \end{aligned}$$

is

Example 23.18

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = (\sqrt{10})^n \begin{pmatrix} c_1 \cos n\theta^* - c_2 \sin n\theta^* \\ -3c_2 \cos n\theta^* - 3c_1 \sin n\theta^* \end{pmatrix}$$

Example 23.19

Example 23.19 Consider the Leslie model of an organism that lives for three years, with death rates $d_1 = 0.2$, $d_2 = 0.6$, and $d_3 = 1$. Suppose that only third year individuals can reproduce, with birth rate $b_3 = 1.6$. The corresponding Leslie matrix

$$\begin{pmatrix} 0 & 0 & 1.6 \\ 0.8 & 0 & 0 \\ 0 & 0.4 & 0 \end{pmatrix}$$

its characteristic polynomial is $p(r) = r^3 - 0.512$, with roots $r = 0.8$ and $-0.4 \pm i\sqrt{0.48}$.

Example 23.19

An eigenvector for $r = 0.8$ is $\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$; an eigenvector is

$$\begin{aligned} r &= -0.4 \mp i\sqrt{0.48} = -0.4(1 \pm i\sqrt{3}) \\ &= -0.4(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}) \end{aligned}$$

is

$$\begin{pmatrix} 1 \mp i\sqrt{3} \\ 1 \pm i\sqrt{3} \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \pm i \begin{pmatrix} -\sqrt{3} \\ +\sqrt{3} \\ 0 \end{pmatrix}$$

Example 23.19

The general solution is

$$\begin{aligned}\begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} &= c_1(0.8)^n \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \\ &+ (-0.4)^n(c_1 \cos n\frac{\pi}{3} - c_2 \sin n\frac{\pi}{3}) \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \\ &- (-0.4)^n(c_2 \cos n\frac{\pi}{3} + c_1 \sin n\frac{\pi}{3}) \begin{pmatrix} -\sqrt{3} \\ +\sqrt{3} \\ 0 \end{pmatrix}\end{aligned}$$

Markov Process

Definition: a stochastic process is a rule which gives the probability that the system (or an individual in this system) will be in state i at time $n+1$, given the probabilities of its being in the various states in previous periods. This probability could depend on the whole previous history of the system. When $P(S_{n+1}|S_n, \cdots, S_1) = P(S_{n+1}|S_n)$, the process is called a Markov process.

Markov Process

Transition matrix or Markov matrix:
$\begin{pmatrix} m_{11} & \cdots & m_{1k} \\ \vdots & \ddots & \vdots \\ m_{k1} & \cdots & m_{kk} \end{pmatrix}$, m_{ij} is called the transition probabilities from state j at time n to state i at time $n+1$. $\sum_i m_{ij} = 1$

Example 23.20

Example 23.20 Consider the employment model of Section 6.2. In this model, each person in the population under study is either employed or unemployed. The two states of this model are employed and unemployed. Let $x^1(n)$ denote the fraction of the study population that is employed at the end of time period n and $x^2(n)$ denote the fraction unemployed.

Example 23.20

Suppose that an employed person has a 90 percent probability of being unemployed next period (and, therefore, a 10 percent probability of being unemployed next period) and that an unemployed person has a 40 percent probability of being employed one period from now (and, therefore, a 60 percent probability of being unemployed).

Example 23.20

The corresponding dynamics are

$$\begin{aligned} x^1(n+1) &= 0.9x^1(n) + 0.4x^2(n) \\ x^2(n+1) &= 0.1x^1(n) + 0.6x^2(n) \end{aligned}$$

or

$$\begin{pmatrix} x^1(n+1) \\ x^2(n+1) \end{pmatrix} = \begin{pmatrix} 0.9 & 0.4 \\ 0.1 & 0.6 \end{pmatrix} \begin{pmatrix} x^1(n) \\ x^2(n) \end{pmatrix}$$

Example 23.20

By the argument of Example 23.4, one eigenvalue of system is $r = 1$. By using the trace result of Theorem 23.9, we conclude that the other eigenvalue is $r = 1.5 - 1.0 = 0.5$.

Example 23.20

To compute the corresponding eigenvectors, we solve

$$\begin{pmatrix} -0.1 & 0.4 \\ 0.1 & -0.4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0.4 & 0.4 \\ 0.1 & 0.1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Example 23.20

We conclude by Theorem 23.6 that the general solution of system is

$$\begin{pmatrix} x_n^1 \\ x_n^2 \end{pmatrix} = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix} \cdot 1^n + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \cdot 0.5^n$$

Example 23.20

Since 1^n and $0.5^n \to 0$ as $n \to \infty$, in the long-run, the solution of the Markov system tends to $w_1 = c_1 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$. Since w_1 should be a probability vector whose components sum to 1, take c_1 to equal the reciprocal 1/5 of sum of the components of w_1. We conclude that the solution of system tends to $\begin{pmatrix} 0.8 \\ -0.2 \end{pmatrix}$ as $n \to \infty$, and that our assumptions lead to a long-run unemployment rate of 20 percent in this community.

Regular Markov matrix

Regular Markov matrix if M^r has only positive entries for some integer r. If $r = 1$,M is called a positive matrix.

Regular Markov matrix

Theorem 23.15 Let M be a regular Markov matrix. Then, (a) 1 is an eigenvalue of M of multiplicity 1 (b) every other eigenvalue r of M satisfies $|r| < 1$ (c) eigenvalue 1 has an eigenvector w_1 with strictly positive components and (d) if we write V_1 for w_1 divided by the sum of its components, then V_1 is a probability vector and each of solution $x(n)$ of $x(n+1) = Mx(n)$ tends to V_1 as $n \to \infty$.

Example 23.21

Example 23.21 Suppose that American families are classified as urban, suburban, or rural and that each year: 20 percent of the urban families move to the suburbs and 5 percent move to rural areas; 2 percent of the suburban dwellers move to urban areas and 8 percent move to rural areas; 10 percent of the rural families move to urban areas and 20 percent move to suburban areas.

Example 23.21

Let $U_n, S_n,$ and R_n denote the fractions of the population classified as urban, suburban, and rural, respectively, n years from now. Then, the data of this problem lead to the Markov system.

$$\begin{pmatrix} U_{n+1} \\ S_{n+1} \\ R_{n+1} \end{pmatrix} = \begin{pmatrix} 0.75 & 0.02 & 0.1 \\ 0.8 & 0.9 & 0.2 \\ 0.05 & 0.08 & 0.7 \end{pmatrix} \begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix}$$

Example 23.21

The eigenvalues are this Markov matrix are 1, 0.7, and 0.65, and the corresponding eigenvectors are

$$\begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix}, \begin{pmatrix} 8 \\ -5 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},$$

where the eigenvector for $r = 1$ has been normalized so that components sum to 1.

Example 23.21

The general solution is

$$\begin{pmatrix} U_n \\ S_n \\ R_n \end{pmatrix} = \begin{pmatrix} 2/15 \\ 10/15 \\ 3/15 \end{pmatrix} + c_2 \begin{pmatrix} 8 \\ -5 \\ -3 \end{pmatrix} 0.7^n + c_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

which converges to (2/15, 10/15, 3/15) as $n \to \infty$. We conclude that in the long-run 2/15 of the population will be living in cities, 2/3 in the suburbs, and 1/5 in rural areas.

Example 23.22

Example 23.22 The eigenvalues of the symmetric matrix

$$B = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{pmatrix}$$

are $r_1 = 2, r_2 = 3, r_3 = 6$. Corresponding eigenvectors are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Example 23.22

As Theorem 23.16 indicates, these vectors are perpendicular to each other. Divide each eigenvector by its length to generate a set of "*normalized eigenvectors*":

$$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

Example 23.22

and make these three orthonormal vectors – vectors which are orthogonal and have length 1 – the columns of the orthogonal matrix

$$\begin{pmatrix} -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{2} \end{pmatrix}$$

Then, $Q^{-1} = Q^T$ and

$$Q^T BQ = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Theorem 23.16 Let A be a $k \times k$ symmetric matrix. Then, (a) all k roots of the characteristic equation $det(A - rI) = 0$ are real numbers. (b) eigenvectors corresponding to distinct eigenvalues are orthogonal; and even if A has multiple eigenvalues, there is a nonsingular matrix P whose columns $w_1, \cdots, w_k$ are eigenvectors of A such that (i) $w_1, \cdots, w_k$ are mutually orthogonal to each other, (ii) $P^{-1} = P^T$

(iii) $P^{-1}AP = P^TAP = \begin{pmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_k \end{pmatrix}$

A matrix P which satisfies the condition $P^{-1} = P^T$, or $P^T P = I$ is called orthogonal matrix.

Example 23.23

Example 23.23 Let's diagonalize a symmetric matrix with nondistinct eigenvalues. Consider the 4X4 symmetric matrix

$$C = \begin{pmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{pmatrix}$$

Example 23.23

By the methods of Section 23.3, the eigenvalues of C are, by inspection 2,2,2 and 6. The set of eigenvectors for 2, the eigenspace of eigenvectors 2, is the three-dimensional nullspace of the matrix

$$C - 2I = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

the space of $\{(u_1, u_2, u_3, u_4) : u_1 + u_2 + u_3 + u_4 = 0\}$.

Example 23.23

Three independent vectors in this eigenspace are

$$v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Example 23.23

In order to construct an orthogonal matrix P so that the product $P^T CP = P^{-1} CP$ is diagonal, we need to find three orthogonal vectors w_1, w_2, w_3 which span the same subspace as the independent vectors v_1, v_2, v_3. The following procedure, called the Gram-Schmidt Orthogonalization Process, will accomplish this task. Let $w_1 = v_1$. Define

$$
\begin{aligned}
w_2 &= v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1, \\
w_3 &= v_3 - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2
\end{aligned}
$$

Example 23.23

The $w_i' s$ so constructed are mutually orthogonal. By construction, w_1, w_2, w_3 span the same space as v_1, v_2, v_3. The application of this process to the eigenvectors v_1, v_2, v_3 yields the orthogonal vectors

$$w_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, w_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix}, w_3 = \begin{pmatrix} -1/3 \\ -1/3 \\ -1/3 \\ 1 \end{pmatrix}$$

Example 23.23

Finally, normalize these three vectors and make them the first three columns of an orthogonal matrix whose fourth column is the normalized eigenvector for $r = 6$:

$$Q = \begin{pmatrix} -1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 1/\sqrt{2} & -1/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 2/\sqrt{6} & -1/\sqrt{12} & 1/2 \\ 0 & 0 & 3/\sqrt{12} & 1/2 \end{pmatrix}$$

Example 23.23

Check that $Q^T = Q^{-1}$ and

$$Q^T C Q = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix}$$

Theorem 23.17

Theorem 23.17 Let A be a symmetric matrix, then
(a) A is positive definite if and only if all the eigenvalues of A are >0
(b) A is negative definite if and only if all the eigenvalues of A are <0
(c) A is positive semidefinite if and only if all the eigenvalues of A are ≥ 0
(d) A is negative semidefinite if and only if all the eigenvalues of A are ≤ 0

Theorem 23.18

Theorem 23.18 Let A be a symmetric matrix. Then, the following are equivalent (a) A is positive definite (b) $\exists$ a nonsigular matrix B s.t. $A = B^T B$ (c) $\exists$ a nonsigular matrix Q s.t. $Q^T A Q = I$.