Mathematical Economics: Lecture 18

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Outline

Definitions

1. $\frac{y(t+1)-y(t)}{y(t)} = r$
2. $y(t+1) = (1+r)y(t)$
3. $\frac{y(t+\Delta t)-y(t)}{y(t)} = r\Delta t$
4. $\frac{y(t+\Delta)-y(t)}{\Delta t} = ry(t)$
5. $\frac{dy}{dt}(t) = ry(t)$

Example 24.1

Example 24.1 A simple example of a differential equation is the equation

$$\dot{y}(t) = 2y(t), \text{ or simply } \dot{y} = 2y. \quad (4)$$

We are asked in (4) to find a function $y(t)$ with the property that taking its derivative is the same as multiplying the function by 2.

Example 24.1

One solution to (4) is $y(t) = e^{2t}$ since its derivative $\dot{y}(t)$ is $2e^{2t} = 2y(t)$. Notice, in this case, that for any constant k, $y(t) = ke^{2t}$ is also a solution of (4). A typical differential equation has a whole one-parameter family of solutions. As was the situation with difference equations, the constant k is determined by the initial value $y(t_0)$ of the variable $y(t)$ under study.

Definitions

Differential equations which describe a relationship between a function of several variables and its partial derivatives are called partial differential equations.

Definitions

An ordinary differential equation is an equation $\dot{y} = F(y, t)$ between the derivative of an unknown function $y(t)$ and an expression $F(y, t)$ involving y and t. If the equation can be written as $\dot{y} = F(y)$, we call it an autonomous or time-independent differential equation. If the equation specifically involves t, we call them nonautonomours or time-dependent.

Example 24.2

Example 24.2 Let's look at another example:$\dot{y}(t) = y^2$. Here, we asked to find the function whose derivative at each t is the same as the square of the function. One solution is $y(t) = -1/t$. To see that this is a solution, take the derivative $\dot{y}$, then compute $(y(t))^2$ and see that you get the same result. (Check this) Once more, there is a one-parameter family of solutions: $y(t) = 1/(k - t)$. Again, check this by computing $\dot{y}$ and y^2 and comparing the two.

Example 24.3

Example 24.3 The equation $\ddot{y} = 3\dot{y} - 2y + 2$ is a second order equation. Its solution is $y(t) = k_1 e^{2t} + k_2 e^t + 1$. (Check) The equation $d^4 y/dt^4 = y$ is a fourth order differential equation, whose solution is

$$y(t) = k_1 \cos t + k_2 \sin t + k_3 e^t + k_4 e^{-t},$$

with four constants of integration.

Explicit Solutions

$\dot{y} = y(100 - 2y)$ has a solution $y(t) = 50$. We call the constant solutions as steady state, stationary solution, stationary point and equilibrium.

Explicit Solutions

1. $\dot{y} = ay \Longrightarrow y(t) = ke^{at}$
2. $\dot{y} = ay + b \Longrightarrow y(t) = -\frac{b}{a} + ke^{at}$
3. $\dot{y} = a(t)y \Longrightarrow y(t) = ke^{\int^t a(s)ds}$
4. $\dot{y} = a(t)y + b(t) \Longrightarrow$
   $y(t) = \left[k + \int^t b(s)e^{-\int^s a(u)du} ds\right] e^{\int^t a(s)ds}$

Explicit Solutions

The above four are called linear differential equations. The first and the third are called homogenous (without b term); the second and fourth with b term are called nonhomogeneous.

Example 24.6

Example 24.6(Derivation of Density Function from Failure Rates) Let f be density function for a continuous random variable $t \geq 0$ and let F be the corresponding distribution function. Think of the random variable t as denoting the lifetime of a mechanical or electrical component. Then,

$$R(t) \equiv 1 - F(t) = Pr\{T > t\},$$

the probability that the component lasts at least t time units, is called the reliability function.

Example 24.6

Given f, F, and R, the failure rate or hazard function Z is defined as

$$Z(t) \equiv \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)}. \quad (13)$$

The function Z can be thought of as the probability that the component will fail tn the next Δt time units, given that it has not failed up to time t, because the latter conditional probability is equal to

$$Pr(t < T \leq t + \Delta t | T > t) = \frac{Pr(t < T \leq t + \Delta t)}{Pr(T > t)}$$

$$f(t)\Delta t$$

Example 24.6

The interesting fact is that one can go the other way: given a hazard function Z, there is a unique corresponding probability density f and distribution F that satisfy (13).
To construct f from Z, note that since $R(t) = 1 - F(t), R' = -F' = -f$. Therefore, (13) can be written as

$$Z(t) = -\frac{R'(t)}{R(t)}, \quad (14)$$

with initial condition $R(0) = 1 - F(0) = 1$.

Example 24.6

Given Z, equation (14) is a linear homogeneous differential equation in R whose solution is

$$R(t) = e^{-\int_0^t Z(s)ds}.$$

Since $f = F' = (1 - R)' = -R'$,

$$f(t) = Z(t)e^{-\int_0^t Z(s)ds}. \quad (15)$$

The impact of going from Z to f is that there is much less structure imposed on Z. One chooses a reasonable failure rate function Z and then (15) determines the corresponding probability density with all the right properties.

Example 24.6

For example, setting $Z(t)$ equal to a constant α implies that the probability of failure is independent of how long the component has been working. The corresponding density function by (15) is

$$f(t) = \alpha e^{-\alpha t},$$

the density function for the exponential distribution. If we add some flexibility and set $Z(t)$ to be a general monomial in t:

$$Z(t) = \alpha\beta t^{\beta-1},$$

Example 24.6

then (15) yields the density function

$$f(t;\alpha,\beta) = (\alpha\beta)t^{\beta-1}e^{-\alpha t^{\beta}}. \quad (16)$$

The random variable with density function (16) is said to have the Weibull distribution.

Example 24.6

- separable equations: $\dot{y} = F(y, t)$ if $F(y, t)$ can be written as a product $F(y, t) = g(y)h(t)$ for some function g and h.
- $\frac{dy}{dt} = g(y)h(t) \Longrightarrow \frac{dy}{g(y)} = h(t)dt \Longrightarrow \int^{y} \frac{dy}{g(y)} = \int^{t} h(t)dt + C$

Example 24.8

Example 24.8 Let $x \to u(x)$ be a utility function for wealth x. The Arrow-Pratt measure of relative risk aversion at wealth x is the expression

$$v(x) = -\frac{u''(x)x}{u'(x)},$$

the elasticity of u' with respect to x.

Example 24.8

In statistical analysis, one would like to work with a utility function u of constant relative risk aversion. Such a u would satisfy the second order differential equation

$$u''(x) = -\frac{u'(x)b}{x}. \quad (18)$$

Let $v(x) = u'(x)$. Then, equation (18) becomes the first order differential equation

$$\frac{dv}{dx} = -\frac{vb}{x},$$

a separable differential equation in v and x.

Example 24.8

Separating the $v's$ from the $x's$ yields

$$\frac{dv}{v} = -b\frac{dx}{x}, \text{ or } \int \frac{dv}{v} = -b\int \frac{dx}{x},$$

whose solution is

$$\ln v = -b(\ln x + C), \text{ or } v = e^{-b\ln x} \cdot e^{-bC} = k_1 e^{\ln x^{-b}} = k$$

Example 24.8

Since $v = u'$

$$u = \int v = \int k_1 x^{-b} = \begin{cases} k_2 + k_1 \ln x & \text{if } b = 1, \\ k_2 + \frac{k_1}{1-b} x^{1-b} & \text{if } b \neq 1, \end{cases}$$

a three-parameter family of functions of constant relative risk aversion. The condition $u' > 0$ requires $k_1 > 0$.

Linear Second Order Equations

$a\ddot{y} + b\dot{y} + cy = 0 \Longrightarrow ar^2 + br + c = 0 \Longrightarrow$
Characteristic equation $r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Linear Second Order Equations

Theorem 24.1 If the characteristic polynomial of the linear second order differential equation has distinct real root r_1, r_2, then the general solution is $y(t) = k_1 e^{r_1 t} + k_2 e^{r_2 t}$

Example 24.12

Example 24.12 Let's solve the initial value problem

$$\ddot{y} - \dot{y} - 2y = 0, \text{ y(0)=3, } \dot{y}(0) = 0$$

The characteristic equation for this problem is $r^2 - r - 2 = 0$. Its roots are $r = 2, -1$. The general solution of the differential equation is

$$y(t) = k_1 e^{2t} + k_2 e^{-1}.$$

Example 24.12

Plug in the two initial values

$$
\begin{aligned}
y(0) &= k_1 + k_2 = 3, \\
\dot{y}(0) &= 2k_1 - k_2 = 0,
\end{aligned}
$$

and solve this system for $k_1 = 1$ and $k_2 = 2$. Therefore, the solution of our initial value problem (28) is $y(t) = e^{2t} + e^{-1}$

Example 24.13

Example 24.13 Let $x \to u(x)$ be a utility function over wealth x. At any wealth level x, the Arrow-Pratt measure of absolute risk aversion $\mu(x)$ equals $-u''(x)/u'(x)$. The function μ is the percent rate of change of u' at x; it is a measure of the concavity of the utility function u.

Example 24.13

To find the utility functions that have constant absolute risk aversion a, we solve the second order differential equation

$$-\frac{u''(x)x}{u'(x)} = a, \text{ or } u''(x) + au'(x) = 0. \quad (29)$$

Example 24.13

Equation (29) is a linear second order differential equation whose characteristic polynomial is $r^2 + ar = 0$, with roots $r = 0, -a$. The general solution of (29) is

$$u(x) = c_1 e^{0x} + c_2 e^{-ax} = c_1 + c_2 e^{-ax},$$

the family affine transformation of e^{-ax}. Note that the condition $u' > 0$ implies that $c_2 < 0$. Check that these $u's$ have absolute risk aversion.

Linear Second Order Equations

Theorem 24.2 If the characteristic polynomial of the linear second order differential equation has equal roots $r_1 = r_2$, then the general solution is $y(t) = k_1 e^{r_1 t} + k_2 t e^{r_2 t}$

Linear Second Order Equations

Theorem 24.3 If the characteristic polynomial of the linear second order differential equation has complex roots $\alpha \pm i\beta$, then the general solution is $y(t) = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t)$

Linear Second Order Equations

$a\ddot{y} + b\dot{y} + cy = g(t) \Longrightarrow$ **Theorem 24.4** Let $y_p(t)$ be any particular solution of the nonhomogeneous differential equation (38). Let $k_1 y_1(t) + k_2 y_2(t)$ be a general solution of the corresponding homogeneous equation $a\ddot{y} + b\dot{y} + cy = 0$. Then, a general solution of (38) is $y(t) = k_1 y_1(t) + k_2 y_2(t) + y_p(t)$.

Example 24.15

Example 24.15 Let's find the general solution of the equation

$$\ddot{y} - 2\dot{y} - 3y = 9t^2. \ (40)$$

The general solution of $\ddot{y} - 2\dot{y} - 3y = 0$ is $y(t) = k_1 e^{3t} + k_2 e^{-t}$. Since the forcing term in (40) is a quadratic in t, we look for a particular solution of (40) that is also a quadratic in t:

$$y_p(t) = At^2 + Bt + C.$$

Example 24.15

Differentiate this candidate solution and plug it into equation (40) to obtain

$$
\begin{aligned}
9t^2 &= \ddot{y}_p - 2\dot{y}_p - 3y_p \\
&= (2A) - 2(2At + B) - 3(At^2 + Bt + C) \\
&= (-3A)t^2 + (-4A - 3B)t + (2A - 2B - 3C)
\end{aligned}
$$

Example 24.15

Since the left$-$ and right$-$hand sides of this equation are equal for all t, the coefficients of each power of t must be equal:

$$\begin{aligned} 9 &= -3A \\ 0 &= -4A - 3B \\ 0 &= 2A - 2B - 3C, \end{aligned}$$

a system whose solution is $A = -3$, $B = 4$, and $C = -14/3$.

Example 24.15

Therefore, a particular solution of (40) is

$$y_p(t) = -3t^2 + 4t - \frac{14}{3},$$

and the general solution of (40) is

$$y(t) = k_1 e^{3t} + k_2 e^{-t} - 3t^2 + 4t - \frac{14}{3}. \text{ (Check.)}$$

Example 24.16

Example 24.16 The general solution of the nonhomogeneous equation

$$\ddot{y} - 2\dot{y} - 3y = 8e^{-t}. \quad (41)$$

is $y(t) = k_1 e^{-t} + k_2 e^{3t} + y_p(t)$, where $y_p(t)$ is a particular solution of (41).

Example 24.16

Given the form of $g(t)$ in (41), a natural candidate for $y_p(t)$ is $y_p = Ae^{-t}$. However, this candidate does not work because the general solution of the homogeneous equation constrains an e^{-t} term. So, we look for a particular solution of the form $y_p(t) = Ate^{-t}$, with an extra t factor.

Example 24.16

Differentiate this candidate twice and plug it into (41):

$$\begin{aligned} 8e^{-t} &= \ddot{y}_p(t) - 2\dot{y}_p(t) - 3y_p(t) \\ &= (Ate^{-t} - 2Ae^{-t}) - 2(-Ate^{-t} + Ae^{-t}) - 3(At \\ &= -4Ae^{-t}, \end{aligned}$$

Therefore, $A = -2$ and $y_p(t) = -2te^{-t}$.

Existence of Solutions

Theorem 24.5 consider the initial value problem $\dot{y} = f(t, y)$ $y(t_0) = y_0$. Suppose that f is a continuous function at the point (t_0, y_0). Then, there exists a C^1 function $y : I \to R^1$ defined on an open interval $I = (t_0 - a, t_0 + a)$ about t_0 such that $y(t_0) = y_0$ and $\dot{y}(t) = f(t, y(t))$ for all $t \in I$, that is, $y(t)$ is a solution of the initial value problem (42). Furthermore, if f is C^1 at (t_0, y_0) , then the solution $y(t)$ is unique; any two solutions of (42) must be equal to each other on the intersection of their domains.

Example 24.17

Example 24.17 Consider the initial value problem

$$\dot{y} = 3y^{2/3}, y(0) = 0. \ (43)$$

Notice that $y^{2/3}$ is a continuous function everywhere, but it is not differentiable at $y = 0$, since its derivative at 0 is finite. This problem falls between the cracks discussed in Theorem 24.5. Theorem 24.5 tells us that this problem has a solution, but it doesn't guarantee that there is only one solution. In fact, $y(t) = 0$ and $y(t) = t^3$ are two solutions of initial value problem (43).

Example 24.18

Example 24.18 Consider the differential equation $\dot{y} = 2ty$. This is an equation that we can solve explicitly as $y = ke^{t^2}$, but let's find this solution geometrically and then compare it with this known solution. At each point (t, y) in the plane, draw a little segment of slope $2ty$.

Example 24.18

For example, in Figure 24.2, we have drawn segments of slope -2 at the points $(1, -1)$ and $(-1, 1)$; segments of slope 2 at the points $(1, 1)$ and $(-1, -1)$; segments of slope 4 at the points $(1, 2)$ and $(-1, -2)$; segments of slope -4 at the points $(-1, 2), (1, -2), (-2, 1)$ and $(2, -1)$; and segments of slope 0 at the points $(0, 1), (0, 0)$ and $(0, -1)$.

Example 24.18

Just by looking at Figure 24.2, we begin to get the picture of functions $y(t)$ with a bowl-shaped graph in the upper half plane and an inverted bowl-shaped graph in the lower half plane.

Example 24.18

However, we need many more segments to get a complete picture. To do this effectively, we need a more efficient process than that of choosing random $(t, y)'s$ and evaluating $\dot{y} = f(t, y)$ at these points. In the previous paragraph, we did choose our points in pairs or triplets with $\dot{y}$ the same at each point of the graph. It is natural to extend this procedure to more than two points at a time and to consider at once all the points where $\dot{y}$ takes on a single value.

Example 24.18

For example, what are all the points (t, y) where the slope $\dot{y}$ is 2? Since $\dot{y} = 2ty$, $\dot{y} = 2$ at all points (t, y) for which $2ty = 2$ or $y = 1/t$. In Figure 24.3, we have sketched lightly the hyperbola $y = 1/t$ and we have drawn little segments of slope 2 along this hyperbola. In this way, we can draw the slope of $y(t)$ for whole curves of points at a time.

Example 24.18

Now, let's do the same for $\dot{y} = 0$, the slope will be zero whenever $2ty = 0$; that is, whenever $t = 0$ or $y = 0-$on both axes. We've drawn little horizontal segments along both axes in Figure 24.4 to mirror the fact that $\dot{y} = 0$ when $t = 0$ or $y = 0$. We have continued this process for slopes $\dot{y} = +2, -1, -2$ in Figure 24.4.

Example 24.18

We now have enough information that we can sketch in some curves which have the appropriate slopes. This is done in Figure 24.5. These curves are indeed the graphs of the family of functions $y = ke^{t^2}$, which we know to be the general solution of $\dot{y} = 2ty$.

Example 24.19

Example 24.19 We will use the method of the previous example to sketch the direction field of $\dot{y} = y^2 + t^2$ efficiently. The level sets of $y^2 + t^2$ are circles $y^2 + t^2 = a$ of radius $\sqrt{a}$. In Figure 24.6, we have drawn a direction field with slope 1/4 on the circle $y^2 + t^2 = 1/4$ of radius 1/2, a field with slope 1 on the circle $y^2 + t^2 = 1$, a field with slope 2 on the circle $y^2 + t^2 = 2$, and a field with slope 4 on the circle $y^2 + t^2 = 4$.

Example 24.19

In Figure 24.7, we have superimposed on the integral field of Figure 24.6, a family of curves which are everywhere tangent to the field. These are the graphs of the solutions of $\dot{y} = y^2 + t^2$.

Example 24.21

Example 24.21 Let's use this method to draw the phase portrait of

$$\dot{y} = y - y^3, \quad (44)$$

We've drawn the graph of $y - y^3$ in Figure 24.12. Since $y - y^3 = y(1 - y)(1 + y)$, the stationary points of (44) occur at $y = 0, 1, -1$. To find the sign of $y - y^3$ in each of the intervals $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$, we just evaluated this function at a point in each of these intervals.

Example 24.21

In the intervals $(-1, 0)$ and $(1, \infty)$, the graph of $y - y^3$ lies below the y$-$axis. This means that $\dot{y} = y - y^3$ is negative there and so $y(t)$ is decreasing$-$a fact that we've marked in each of these intervals with an arrow pointing to the left. Similarly, in the intervals $(-\infty, -1)$ and $(0, 1)$, the graph of $y - y^3$ lies below the y$-$axis. This means that $\dot{y} = y - y^3$ is positive and $y(t)$ is increasing$-$a fact that we've marked in these intervals in Figure 24.12 with arrows pointing to the right.

Example 24.21

We can easily read the evolution of differential equation (44) from the phase portrait in Figure 24.12. If one starts to the left of 0, the system tends to the steady state $y = -1$. If one starts anywhere to the right of $y = 0$, the system tends to the steady state $y = 1$. The unstable equilibrium at $y = 0$ is the boundary (or separatrix) between the region of attraction of $y = -1$ and that of $y = +1$.

Example 24.22

Example 24.22 Let's add a small complication to the differential equation (44) and consider the equation

$$\dot{y} = e^{y}(y - y^{3}), \quad (45)$$

This equation will surely not have an illuminating closed form solution, if it has a closed form solution at all. To draw its phase portrait, we need to draw the graph of $f(y) = e^{y}(y - y^{3})$.

Example 24.22

However, we are really only interested in where f is positive and where f is negative. Since the e^y factor is always positive, $f(y)$ will have the same sign as $y - y^3$ for any y. This implies that equation (45) has the same phase portrait as equation (44) in the previous example. The only difference is in the speed of the motion; for example, $y(t)$ will move to $+\infty$ much more quickly for (45) than it will for (44). The time factor is hidden when we draw the phase portrait.

Phase portraits and equilibria on R^1

Theorem 24.6 Let y_0 be a rest point of the C^1 differential equation $\dot{y} = f(y)$ on the line; so $f(y_0) = 0$. If $f'(y_0) < 0$, then y_0 is an asymptotically stable equilibrium. If $f'(y_0) > 0$, then y_0 is an unstable equilibrium.