Undergraduate Mathematical Economics Lecture 1

Yu Ren

WISE, Xiamen University

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Outline

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- mathematical techniques used in economics courses
- *Mathematics for Economists* by Simon and Blume

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- 15 weeks, including one midterm exam
- grading policy
  - quiz: 30 %
  - midterm exam:30%
  - final exam:40%

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Office hours

- my office hours: 2:30-4:00pm Tuesday
- TA office hours: TBA
- any courses-related questions can be asked

Some Questions

1. Why economics need mathematics?
2. What is the focus of this course?
3. How to achieve good performances in this course

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Linear Algebra

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Chapter 6, 7, 8, 9

Linear Equation

- an equation is **linear** if

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

- a_i: parameters, x_i: variables ($i = 1, 2, \cdots, n$)

Linear Equation

For example

- $x_1 + 2x_2 = 3$
- $2x_1 - 3x_2 = 8$

Linear Systems

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
\vdots & + & \vdots & + & \cdots & + & \vdots & = & \vdots \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

Example: Tax Benefits

- Before tax profits: 100,000
- 10% of after-tax profits to Red Cross
- state tax of 5% of its profits after Red Cross donation
- federal tax of 40% of its profits after donation and state tax

Example: Tax Benefits

- Question: How much does the company pay in state taxes, federal taxes and Red Cross?

Example: Answer

C: (Red Cross); S: (State taxes); F: (Federal taxes)

$$C = (100,000 - S - F) * 0.1 \quad (1)$$
$$S = (100,000 - C) * 0.05 \quad (2)$$
$$F = (100,000 - C - S) * 0.4 \quad (3)$$

General Questions

Questions:

1. Does a solution exist?
2. How many solutions are there?
3. Is there an efficient algorithm that computes actual solutions?

General Questions

method to find the answers

1. substitutions
2. elimination of variables
3. matrix methods

Substitution

- Taught in high school class
- Steps:
  1. m equations, n variables
  2. solve x_n in terms of the other variables
  3. substitute this expression for x_n into the other equations
  4. a new system of $m - 1$ equations and $n - 1$ variables
  5. repeat the process

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Elimination

- Example (Page 126):

$$\begin{array}{rcrcrcr} x_1 & - & 0.4x_2 & - & 0.3x_3 & = & 130 \\ -0.2x_1 & + & 0.88x_2 & - & 0.14x_3 & = & 74 \\ -0.5x_1 & - & 0.2x_2 & + & 0.95x_3 & = & 95 \end{array}$$

Matrix Methods: Elementary Row Operations

- augmented matrix: add on a column corresponding to the right-hand side in system

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{pmatrix}$$

Matrix Methods: Elementary Row Operations

- interchange two rows of a matrix
- change a row by adding to it a multiple of another row
- multiply each element in a row by the same nonzero number

Definitions

- **leading zeros**: a row of a matrix is said to have k leading zeros if the first k elements of the row are **all** zeros and the $k+1$ element of the row is not zero

Definitions

- **row echelon form**: a matrix is in row echelon form if each row has more leading zeros than the row preceding it. Example:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix}$$

- row echelon form can be obtained by elementary row operations

Rank

- **Rank**: the rank of a matrix is the number of nonzero rows in its row echelon form.
- a matrix $\rightarrow$ row echelon form $\rightarrow$ rank

Rank

- Let A be the coefficient matrix of some linear systems and let $\hat{A}$ be the corresponding augmented matrix. Then this systems has a solution if and only if rank($\hat{A}$)=rank(A).

Matrix Algebra: Addition

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} + \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} + b_{11} & \cdots & a_{1n} + b_{1n} \\ \vdots & a_{ij} + b_{ij} & \vdots \\ a_{k1} + b_{k1} & \cdots & a_{kn} + b_{kn} \end{pmatrix}$$

Matrix Algebra: Subtraction

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} - \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & b_{ij} & \vdots \\ b_{k1} & \cdots & b_{kn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11} - b_{11} & \cdots & a_{1n} - b_{1n} \\ \vdots & a_{ij} - b_{ij} & \vdots \\ a_{k1} - b_{k1} & \cdots & a_{kn} - b_{kn} \end{pmatrix}$$

Matrix Algebra: scalar multiplication

$$r\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} = \begin{pmatrix} ra_{11} & \cdots & ra_{1n} \\ \vdots & ra_{ij} & \vdots \\ ra_{k1} & \cdots & ra_{kn} \end{pmatrix}$$

Matrix Algebra: matrix multiplication

We can define the matrix product AB if and only if **number of column of A = number of rows of B .**

(i, j) entry of AB is

$$\begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{im} \end{pmatrix} \begin{pmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{mj} \end{pmatrix} = \sum_{h=1}^{m} a_{ih} b_{hj}$$

Laws of Matrix Algebra

- Associative Laws:
  $(A+B)+C = A+(B+C)$; $(AB)C = A(BC)$
- Commutative Law for addition:
  $A+B = B+A$
- Distributive Laws: $A(B+C) = AB+AC$;
  $(A+B)C = AC+BC$

Transpose

- transpose:

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & a_{ij} & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{k1} \\ a_{12} & a_{22} & \cdots & a_{k2} \\ \vdots & \vdots & a_{ji} & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{kn} \end{pmatrix}$$

Transpose

- $(A+B)^T = A^T + B^T$; $(A-B)^T = A^T - B^T$; $(A^T)^T = A$; $(rA)^T = rA^T$; $(AB)^T = B^T A^T$

Special Matrices

- Page 160
- Square Matrix; Column Matrix; Row Matrix; Diagonal Matrix; Upper-Triangular Matrix; Lower-Triangular Matrix; Symmetric Matrix; Permutation Matrix
- Idempotent Matrix; Nonsingular Matrix

Inverse

- Let A be a $n \times n$ matrix. Matrix B is an inverse of A if $AB = BA = I$. If the matrix B exists, we say A is **invertible**.

Inverse

- **Theorem 8.5**: An $n \times n$ matrix can have at most one inverse.
- $(A^{-1})^{-1} = A$; $(A^T)^{-1} = (A^{-1})^T$; AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$
- How to find the inverse matrix? Hold this question for a while.

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Determinants

- $det(a) = a$
- $det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21}$

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Minor and Determinant

- Let A be an $n \times n$ matrix. Let A_{ij} be the $(n-1) \times (n-1)$ submatrix obtained by deleting row i and column j from A. The number $M_{ij} = det(A_{ij})$ is called the (i, j) **minor** of A and the scalar $C_{ij} = (-1)^{i+j} M_{ij}$ is called the (i, j) **cofactor** of A.

Minor and Determinant

- Determinant of an $n \times n$ matrix A is given by

$$det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- Example 9.2 (Page 192)

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- Example 9.2 (Page 192)

Uses of the Determinant

- For any $n \times n$ matrix A, let C_{ij} denotes the (i, j)th cofactor of A. The $n \times n$ matrix whose (j, i) entry is C_{ij} is called the adjoint of A, denoted by $adj(A)$.

Uses of the Determinant

- $A^{-1} = \frac{1}{det(A)} adj(A)$

Uses of the Determinant

- (Cramer's Rule) the unique solution $x = (x_1, \cdots, x_n)$ of the $n \times n$ system $Ax = b$ is $x_i = \frac{det(B_i)}{det(A)}$, where B_i is the matrix A with the right-hand side b replacing the ith column of A
- Example 9.3 (Page 195)

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- Example 9.3 (Page 195)

Some Properties of Determinant

- $det(A^T) = det(A)$
- $det(AB) = (det(A))(det(B))$
- $det(A + B) \neq det(A) + det(B)$