Mathematical Economics: Lecture 2

Yu Ren

WISE, Xiamen University

September 25, 2012

Outline

1 Chapter 2

Number Line

- The number line, origin (Figure 2.1 Page 11)

Number Line

- Interval

  $(a, b) = \{x \in R^1 : a < x < b\}$
  $[a, b] = \{x \in R^1 : a \leq x \leq b\}$
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Function on R^1

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Function on R^1

- $x = 0 \rightarrow f(x) = 0.5$
  $x = 3.25 \rightarrow f(x) = 0.98$
  $x = \pi \rightarrow f(x) = \sqrt{2}$
  Is $f(x)$ a function about x ?

Function on R^1

- Domain: the set of numbers at which the function is defined
  Example: $f(x) = \frac{1}{x}$

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Function on R^1

- x: independent variable, exogenous variable
- $y = f(x)$: dependent variable, endogenous variable

Function Categories

- monomials: $f_1(x) = 4x^3$, $f_1(x) = 5x^2$, $f_1(x) = 2x$
- $f(x) = ax^b$
  b: positive integer

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Function Categories

- rational function $f(x) = \frac{polynomial}{polynomial}$,

Function Categories

- exponential function $f(x) = a^x$,

Function Categories

- trigonometric function
- $f(x) = F(sin(x), cos(x), \cdots),$

Function Categories

- increasing function, decreasing function

Minimum

- **local minimum**: a minimum within a local neighborhood
- For example: if x_1 is a local minimum of $f(x)$, then $f(x_1)$ is the lowest value within $(x_1 - \epsilon, x_1 + \epsilon)$, where ϵ is a small positive number

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Linear Function

- linear function: $f(x) = mx + b$
- m is the slope, b is the intercept

Linear Function

- **Example 2.2** The slope of the line joining the points (4,6) and (0,7) is

$$m = \frac{7-6}{0-4} = -\frac{1}{4}.$$

Economic Interpretation: Marginal

- Slope in economics: marginal cost, marginal utility, marginal product
- **marginal cost** is the change in total cost that arises when the quantity produced changes by one unit.
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Slope of Nonlinear Function

- nonlinear function
- derivative: $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$

Slope of Nonlinear Function

- computing rules: **Theorem 2.4**:
  - $(f \pm g)'(x_0) = f'(x_0) \pm g'(x_0)$
  - $kf'(x_0) = k(f'(x_0))$
  - $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$
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Example

Example 2.5:
For $f(x) = x^2$, in order to prove that $f'(3) = 6$, we need to show that

$$\frac{(3 + h_n)^2 - 3^2}{h_n} \to 6, \text{ as } h_n \to 0$$

for every sequence $\{h_n\}$ which approaches zero.

Example

For any h,

$$\frac{(3+h)^2-3^2}{h}=\frac{9+6h+h^2-9}{h}=\frac{h(6+h)}{h}=6{+}h$$

which clearly converges to 6 as $h \to 0$.
$f'(3)=6$.

Differentiability and Continuity

- differentiable: $\lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists and same $\forall$ h
- Is $f(x) = |x|$ differentiable at $x_0 = 0$?

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Differentiability and Continuity

- continuous: $f(x_0) = \lim_{h \to 0} f(x_0 + h)$
- when the limit of x_0 exists, f may or may not be continuous at x_0
- $f(x) = x$ when $(-\infty, 3) \cup (3, +\infty)$ and $f(3) = 9$. $f(x)$ is not continuous at 3.

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Higher-Order Derivatives

- $f''(x_0)$ or $\frac{d}{dx}(\frac{df}{dx})(x_0) = \frac{d^2f}{dx^2}(x_0)$
- **Example 2.9**: the derivative of the function $f(x) = x^3 + 3x^2 + 3x + 1$ is
  $f'(x) = 3x^2 + 6x + 3$
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Approximation by Differentials

- $f'(x_0) = \frac{dy}{dx} \approx \frac{\Delta y}{\Delta x}$
- $\Delta y = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0)\Delta x$
- $f(x_0 + \Delta x) \approx f(x_0) + f'(x_0)\Delta x$
- Figure 2.15 (page 37)

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Example

Example 2.11

- Production function $F(x) = \frac{1}{2}\sqrt{x}$
- 100 units of labor input x
- its output is 5 units.
- The derivative of the production F at $x = 100$,

$$F'(100) = \frac{1}{4}100^{-1/2} = \frac{1}{40} = 0.025,$$

Example

- $F'(100)$ is a good measure of the marginal product of labor
- the actual increase in output is $F(101) - F(100) = 0.02494\cdots$, pretty close to 0.025.