Mathematical Economics: Lecture 4

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Outline

New Section

Chapter 4

Composite Functions

- $f(x) = h(g(x))$, $f(x) = (h \circ g)(x)$

Composite Functions

- **Example 4.1** $g(x) = x^2$, $h(x) = x + 4$,
  $(h \circ g)(x) = x^2 + 4$
  $(g \circ h)(x) = (x + 4)^2$.
  Note that $h \circ g \neq g \circ h$

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Example in Economics

Example 4.2

- firm's production function $y = f(L)$
- L is the amount of labor input
- profit function $P(L) \equiv \Pi(f(L)) = (\Pi \circ f)(L)$
- $\Pi(y) = -y^4 + 6y^2 - 5$ and $f(L) = 5L^{2/3}$, (1)

Example in Economics

Example 4.2

$$
\begin{aligned}
P(L) &= \Pi(f(L)) \\
&= -(5L^{2/3})^4 + 6(5L^{2/3})^2 - 5 \\
&= -625L^{8/3} + 150L^{4/3} - 5.
\end{aligned}
$$

Differentiating Composite Function

- $\frac{d}{dx}(h(g(x))) = h'(g(x))g'(x)$

Differentiating Composite Function

- **Example 4.4** apply the Chain Rule to Example 4.2,

$$P'(L) = \frac{d}{dL}(\Pi(f(L))) = \Pi'(f(L)) \cdot f'(L)$$
$$= (-4(5L^{2/3})^3 + 12(5L^{2/3})) \cdot (\frac{10}{3}L^{-1/3})$$
$$= -\frac{5000}{3}L^{5/3} + 200L^{1/3}.$$

Differentiating Composite Function

directly take the derivative,

$$(-625L^{8/3}+150L^{4/3}-5)' = -\frac{5000}{3}L^{5/3}+200L^{1/3}$$

Inverse Functions

- $f : E_1 \to R^1$, $g : E_2 \to R^1$
- If $g(z)$ is the inverse function of $f(x)$
  $g(f(x)) = x$ for all $x \in E_1$
  $f(g(z)) = z$ for all $z \in E_2$

Inverse Functions

Example 4.7

$$
\begin{aligned}
f(x) &= 2x \ \& \ g(y) = \frac{1}{2}y, \\
f(x) &= x^2 \ \& \ g(y) = \sqrt{y}, \ \text{for } x, y \geq 0 \\
f(x) &= x^3 \ \& \ g(y) = y^{1/3}, \\
f(x) &= \frac{x-1}{x+1} \ \& \ g(y) = \frac{1+y}{1-y}, \\
f(x) &= \frac{1}{x} \ \& \ g(y) = \frac{1}{y}.
\end{aligned}
$$

Inverse Functions: Note

- in order for f to have an inverse g, f cannot assign the same point to two different points in its domain.
- $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
- $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$
- "one-to-one" or "**injective**"

Inverse Functions: Note

Example 4.8 $f(x) = x^2$ is not one-to-one, but restrict the domain of f to be the nonnegative numbers $[0, \infty)$, then it has a well-defined inverse $g(y) = \sqrt{y}$. See Figure 4.3.

Inverse Functions: Theorem

Theorem 4.2: A function f defined on an interval E in R^1 has a well-defined inverse on the interval $f(E)$ if $f'(x) > 0$ for all $x \in E$ or $f'(x) < 0$ for all $x \in E$

Derivatives of the inverse function

Theorem 4.3: Let f be a C^1 function defined on the interval I in R^1. If $f'(x) \neq 0$ for all $x \in I$, then

- f is invertible on I
- its inverse g is a C^1 function on the interval $f(I)$
- for all z in the domain of the inverse function g

$$g'(z) = \frac{1}{f'(g(z))}$$

Examples

Example 4.11 $f(x) = \frac{x-1}{x+1}$ and x=2, then f(2)=1/3 and g(1/3)=2.

$$f'(x) = \frac{2}{(x+1)^2} \text{ and } f'(2) = \frac{2}{9},$$

$$g'(\frac{1}{3}) = \frac{1}{f'(2)} = \frac{1}{2/9} = \frac{9}{2}.$$

Examples

check directly:

$$g(y) = \frac{1+y}{1-y},$$
$$g'(y) = \frac{2}{(1-y)^2},$$
$$g'(\frac{1}{3}) = \frac{2}{4/9} = \frac{9}{2}.$$

New Section

Application to Economics

Production

- $y = f(q)$, q: input; y: output
- continuous or C^2
- increasing $f'(q) > 0$
- for some $a \geq 0$, $f''(q) > 0$ for $q \in [0, a)$ and $f''(q) < 0$ for $q > a$

Production

- two-parameter family $y = kq^b$ $k > 0$ $b > 0$
- $f = \sqrt{q}$

Cost Function

- $y = C(x)$, x: output; y: cost
- C^1
- increasing $C'(x) > 0$
- $C'(x)$: marginal cost $MC(x)$
- average cost: $AC(x) = \frac{C(x)}{x}$

Cost Function

- $MC > AC$, then AC is increasing
- $MC < AC$, then AC is decreasing
- at an interior minimum of AC, $AC = MC$

Revenue and Profit Function

- $R(x)$; revenue function, indicating how much money a firm receives for selling x units of its output
- increasing in x
- profit function: $\prod(x) = R(x) - C(x)$
- optimal output: $MR(x^*) = MC(x^*)$

Demand Function and Elasticity

- demand function: $x = F(p)$ expresses x consumed in terms of price level p
- inverse demand function: $p = G(x)$
- price elasticity of demand: $\varepsilon = \frac{\Delta x}{x} / \frac{\Delta p}{p}$
- $\varepsilon \in (-1, 0)$: inelastic
- $\varepsilon \in (-\infty, -1)$: elastic
- $\varepsilon = -1$: unit elastic

Present Value and Annuities

- put A dollars into an account which compounds interest continuously at rate r, after t years
- $B = Ae^{rt}$
- $A(1+r)^t$, $A(1+r/2)^{2t}$, $A(1+r/4)^{4t}$

Present Value and Annuities

- $\lim_{n \to +\infty} A(1 + r/n)^{tn} = Ae^{rt}$
- present value: $A = Be^{-rt}$
- annuity: a sequence of equal payments at regular intervals over a specified period of time