Mathematical Economics: Lecture 6

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Outline

New Section

Chapter 11: Linear Independence

Linear Independence

- Vector $V_1, \cdots, V_k$ in R^n are linear dependent if and only if $\exists\ c_1, \cdots, c_k$ not **all** zero, s.t. $c_1 V_1 + \cdots + c_k V_k = 0$.

Linear Independence

- Vector $V_1, \cdots, V_k$ in R^n are linear independent if and only if $c_1 V_1 + \cdots + c_k V_k = 0$ implies $c_1 = c_2 = \cdots = c_k = 0$

Linear Independence; Example

- **Example 11.2** The vectors

$$w_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \; w_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \text{ and } w_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$$

are linearly dependent in R^3, since

$$1 \cdot \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - 2 \cdot \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix} + 1 \cdot \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

as can easily be verified.

Linear Independence

- **Theorem 11.1**: Vectors $v_1, \cdots, v_k$ in R^n are linearly dependent if and only if the linear system $A = (v_1, \cdots, v_k), Ac = 0$ has nonzero solution $(c_1, \cdots, c_k)$.

Linear Independence

- **Theorem 11.2**: A set of n vectors $v_1, \cdots, v_n$ in R^n is linearly independent if and only if $det(v_1, \cdots, v_n) \neq 0$

Linear Independence

- **Theorem 11.3** If $k > n$, any set of k vectors in R^n is linearly dependent.

Spanning Sets

- the line generated or spanned by V: $L[V] \equiv \{rV : r \in R^1\}$.
- Figure 11.1 (page 238)

Spanning Sets

- Set generated or spanned by $V_1, \cdots, V_k$: $L[V_1, \cdots, V_k] \equiv \{c_1 V_1 + \cdots + c_k V_k : c_1 \cdots, c_k \in R^1\}$.

Spanning Sets: Example

- **Example 11.4** The x_1x_2-plane in R^3 is the span of the unit vectors $e_1 = (1, 0, 0)$ and $e_2 = (0, 1, 0)$, because any vector (a, b, 0) in this plane can be written as

$$\begin{pmatrix} a \\ b \\ 0 \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Spanning R^k

- any linear independent vector $V_1, \cdots, V_k$ can span R^k. This means any vector $U \in R^k$, $\exists\, c_1, \cdots, c_k$, s.t. $c_1 V_1 + \cdots + c_k V_k = U$.
- To judge whether B belongs to the space spanned by $\{V_1, \cdots, V_k\}$, check whether $Vc = B$ has a solution c. (Theorem 11.4)
- A set of vectors that span R^n must contain at least n vectors. (Theorem 11.6)

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Spanning R^k: Example

Example 11.1 The vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

in R^n are linearly independent

Spanning R^k: Example

because if $c_1, ..., c_n$ are scalars such that $c_1 e_1 + c_2 e_2 + \cdots + c_n e_n = 0$,

$$c_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix} + \ldots + c_n \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

The last vector equation implies that $c_1 = c_2 = c_n = 0$.

Basis and dimension in R^n

Definition: Let $V_1, \cdots, V_k$ be a fixed set of k vectors in R^n. let V be the set $L[V_1, \cdots, V_k]$ spanned by $V_1, \cdots, V_k$. Then if $V_1, \cdots, V_k$ are linearly independent, $V_1, \cdots, V_k$ is called a basis of V. More generally, let $W_1, \cdots, W_k$ be a collection of vectors in V. Then $W_1, \cdots, W_k$ forms a basis if

- (a) $W_1, \cdots, W_k$ span V and
- (b) $W_1, \cdots, W_k$ are linearly independent.

Basis and dimension in R^n: Example

Example 11.7

- $v_1 = (1, 1, 1)$, $v_2 = (1, -1, -1)$, and $v_3 = (2, 0, 0)$
- $v_3 = v_1 + v_2$
- linear combination of v_1, v_2, and v_3 = linear combination of just v_1, and v_2

Basis and dimension in R^n: Example

$$
\begin{aligned}
w = & \quad av_1 + bv_2 + cv_3 \\
= & \quad av_1 + bv_2 + c(v_1 + v_2) \\
= & \quad (a + c)v_1 + (b + c)v_2.
\end{aligned}
$$

The set $\{v_1, v_2\}$ is a more efficient spanning set than is the set $\{v_1, v_2, v_3\}$.

Basis and dimension in R^n: Example

- **Example 11.8** We conclude from Examples 11.1 and 11.5 that the unit vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

  form a basis of R^n. Since this is such a natural basis, it is called the canonical basis of R^n.

Basis and dimension in R^n

- $V_1, \cdots, V_k$ are linearly independent $\Longleftrightarrow$ $V_1, \cdots, V_k$ span R^k. $\Longleftrightarrow$ $V_1, \cdots, V_k$ form a basis of R^k $\Longleftrightarrow$ the determinant of $[V_1, \cdots, V_k]$ is nonzero. (Theorem 11.8)
- we call the number of basis of a space as the dimension.