Mathematical Economics: Lecture 7

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Outline

New Section

Chapter 12: Limits and Open Sets

Sequences of Real Numbers

- A sequence of real numbers: a mapping from **all** natural numbers to real numbers.
  - There are infinite number of entries in a sequence
  - May not have an explicit function to describe the mapping

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Example

Example 12.1 Some examples of a sequence of real numbers are:

- $\{1, 2, 3, 4, \ldots\}$
- $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$
- $\{1, \frac{1}{2}, 4, \frac{1}{8}, 16, \ldots\}$
- $\{0, -\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, \ldots\}$
- $\{-1, 1, -1, 1, -1, \ldots\}$
- $\{\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots\}$
- $\{3.1, 3.14, 3.141, 3.1415, \ldots\}$
- $\{1, 4, 1, 5, 9, \ldots\}$

Limit of Sequences

Limit of a sequence

- Intuitive Definition: a number to which the entries of the sequences approach *arbitrarily* close. (How to define *arbitrarily* ?)

Limit of Sequences

Limit of a sequence

- Mathematical Definition: for any $\{x_n\}$, r is the limit of this sequence if for any small $\varepsilon > 0$, $\exists$ N, s.t. for **all** $n \geq N$, $|x_n - r| < \varepsilon$.
- $|x_n - r| < \varepsilon \Longleftrightarrow x_n \in I_\varepsilon(r)$
- $I_\varepsilon(r) = \{s \in R : |s - r| < \varepsilon\}$

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Limit of Sequences: Example

Example 12.2 Here are three more sequences which converge to 0:

$$1, 0, \frac{1}{2}, 0, \frac{1}{3}, 0, \ldots$$

$$1, -\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, \ldots$$

$$\frac{1}{1}, \frac{3}{1}, \frac{1}{2}, \frac{3}{2}, \frac{1}{3}, \frac{3}{3}, \frac{1}{4}, \ldots$$

Sequences of Real Numbers

Accumulation Point: if for any positive ε, there are infinitely elements in $I_\varepsilon(r)$

- different with the limit
- only need infinitely elements not all x_n $(n \geq N)$

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Sequences of Real Numbers

Properties of Limits

- **Theorem 12.1**: A sequence can have at most one limit
- **Theorem 12.2** If $x_n \to x$ and $y_n \to y$, then $x_n + y_n \to x + y$
- **Theorem 12.3** If $x_n \to x$ and $y_n \to y$, then $x_n \times y_n \to xy$

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Sequences of Real Numbers

Subsequence: $\{y_j\}$ is a subsequence of $\{x_i\}$ if $\exists$ an infinite increasing set of natural number $\{n_j\}$ s.t. $y_1 = x_{n_1}$, $y_2 = x_{n_2}$,$y_3 = x_{n_3}$, for example $\{1, -1, 1, -1, \cdots\}$ has two subsequences: $\{1, 1, \cdots\}$ $\{-1, -1, \cdots\}$ and so on.

Sequence in R^m

- Definition: $\{x_i\}$, $x_i \in R^m$
- Euclidean metric in R^m:

$$d(x_i, x_j) = ||x_i - x_j|| = \sqrt{(x_{i1} - x_{j1})^2 + \cdots + (x_{im} - x_{jm})^2}$$

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Sequence in R^m

- ε ball about r:
  $B_\varepsilon(r) \equiv \{x \in R^m : ||x - r|| < \varepsilon\}$
- $\{x_i\}$ is said to converge to the vector: for any $\varepsilon > 0$, $\exists$ N, s.t. for any $n \geq N$, $x_n \in B_\varepsilon(x)$

Sequence in R^m

Properties

- **Theorem 12.5** $x_n \to x$ if and only if $x_{in} \to x_i$ for all $i = 1, 2, \cdots, m$.
- **Theorem 12.6** $x_n \to x$, $y_n \to y$ and $c_n \to c$ then $c_n x_n + y_n \to cx + y$.

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Sequence in R^m

Accumulation Point: if for any positive ε, there are infinitely elements in $B_\varepsilon(r)$

Open Sets

- Open sets: A set S in R^m is open if for each $x \in S$, $\exists\ \varepsilon > 0$ s.t. $B_\varepsilon(x) \subset S$.
- Interior: $S \subseteq R^m$. The union of all open sets **contained** in S is called the interior of S, denoted by *intS*

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Closed sets

- Closed sets: A set S in R^m is closed if whenever $\{x_n\}$ is a convergent sequence completely contained in S, its limit is also contained in S.
- Closure: $S \subseteq R^m$. The intersection of all closed sets **containing** S is called the closure of S, denoted by clS.

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Compact sets

- Boundary: x is in the boundary of S if every open ball about x contains both points in S and points in S^c.
- Bounded: a set S in R^m is bounded if $\exists$ a number B st $||x|| \leq B$ for all $x \in S$.
- Compact sets: A set S in R^m is compact if and only if it is both closed and bounded.

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Open Sets, Closed sets and compact sets

- $S = \{x \in R^2 : d(x, \mathbf{1}) \leq 2\}$
- $S = \{x \in R^2 : d(x, \mathbf{1}) < 2\}$
- $S = \{x \in R^2 : 1 \leq d(x, \mathbf{1}) \leq 2\}$
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Properties

- **Theorem 12.7**: open balls are open sets
- **Theorem 12.8**: any union of open sets is open; the **finite** intersection of open sets is open.

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- **Theorem 12.9**: S is closed if and only if $R^m - S$ is open.
- **Theorem 12.10**: any intersection of closed sets is closed; the finite union of closed sets is closed.

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Properties

- **Theorem 12.13**: Any sequence contained in the closed and bounded interval $[0, 1]$ has a convergent subsequence
- **Theorem 12.14**: Let C be a compact subset in R^n and let $\{x_n\}$ be any sequence in C. Then, $\{x_n\}$ has a convergent subsequence whose limit lies in C.

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Properties

- **Theorem 12.11**: $x \in clS$ if and only if there is x_n in S converging to x.
- **Theorem 12.12**: boundary = $clS \cap clS^c$

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