Mathematical Economics: Lecture 8

Yu Ren

WISE, Xiamen University

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Outline

New Section

Chapter 13: Function of Several Variables

Functions between Euclidean Spaces

A function $f : A \to B$ a set A to a set B is a rule that assigns to **each** object in A, **one and only one** object in B.

- Domain: the set A of elements on which f is defined
- Target: the set B in which f takes its values
- Image: $y = f(x) \in B$
- Preimage: the preimage of V is $f^{-1}(V) = \{a \in A : f(a) \in V\}$

Functions between Euclidean Spaces

A function $f : A \rightarrow B$ a set A to a set B is a rule that assigns to **each** object in A, **one and only one** object in B.

- Surjective: the whole target space of f is the image of f
- Injective: $f : A \rightarrow B$ is one-to-one
- Inverse function: $f^{-1} : f(C) \rightarrow C$
- Composition: $f : A \rightarrow B$ and $g : C \rightarrow D$. The composition of f with g, $(g \circ f)(x) = g(f(x))$.

Example

Example 13.1 the function $f : R^2 \to R^1$ defined by $f(x, y) = x^2 + y^2$.
the domain of f is all of R^2,
the target space of f is R^1,
the image of f is the set of all nonnegative real numbers.

Example

Example 13.2 The domain of the function $g(x) = 1/x$ is R^1 - $\{0\}$, all real numbers except 0.
its image is R^1 - $\{0\}$, too.

Vocabulary

- Onto: if for each element $b \in B$ there is an element $a \in A$ s.t. $b = f(a)$, in other words, if the whole target space of f is the image of f, we say f maps A onto B or f is surjective
- inverse: $f^{-1} : f(C) \to C$

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Vocabulary

- composition: let $f : A \to B$ and $g : C \to D$ be two functions. Suppose that B, the image of f, is a subset of C , the domain of g. Then, the compostition of f with g, $g \circ f : A \to D$ is defined as the function $(g \circ f)(x) = g(f(x))$ for all $x \in A$
- Figure 13.20

Vocabulary: Example

Example 13.9 function $f(x_1, x_2) = x_1^2 + x_2^2$, its target space is R^1; its image is the set of all nonnegative real numbers. Since the two are not the same, f is not onto. Neither is f one-to-one since $f(1, 0) = f(0, 1) = 1$.

Vocabulary: Example

Example 13.10 the target space of the function $f(x) = \frac{1}{x}$ is R^1, but its image is R^1 - 0. f is not onto, and is a one-to-one function.

Vocabulary: Example

Example 13.11 consider the functions f and g from R^1 to R^1 defined by $f(x) = 2x$ and $g(x) = 2x - 1$. For both of these, the domain and image are all of R^1. Both are one-to-one maps of R^1 onto R^1. Then their inverses are given by

$$f^{-1}(y) = \frac{1}{2}y \text{ and } g^{-1}(y) = \frac{1}{2}(y + 1).$$

Vocabulary: Example

Example 13.12 the function $h(x) = \sin x^2$ is the composition of $f(x) = x^2$ with $g(x) = \sin x$.
the function $h(x) = (x + 4)^3$ is the composition of $f(x) = x + 4$ with $g(x) = x^3$.
the function $h(x) = \sqrt{x^2 + y}$ is the composition of $f(x, y) = x^2 + y$ with $g(z) = \sqrt{z}$.

Utility Function

A utility function is a function which assigns a number of $u(x_1, \cdots, x_k)$ to each commodity bundle.

Some regularly used functions

- Linear: $q = a_1x_1 + a_2x_2$
- Cobb-Douglas: $q = kx_1^{b_1}x_2^{b_2}$
- Liontiff: $q = \min\{\frac{x_1}{c_1}, \frac{x_2}{c_2}\}$
- CES: $q = k(c_1x_1^{-a} + c_2x_2^{-a})^{-b/a}$

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Level Curve

- Level curve: for each (x, y) we once evaluate $f(x, y)$ to obtain z_0. Then sketch the locus in the $xy - plane$ of all other (x, y) pairs for which f has the same value of z_0. Example 13.3 Figure 13.7.
- Level sets for production functions are called isoquants. Figure 13.10.
- Level curves of a utility function are called indifference curves.

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Level Curve: Example

Example 13.3 $f(x, y) = x^2 + y^2$ start with (0, 1), $f = 1 \Rightarrow$ the set $\{(x, y) : x^2 + y^2 = 1\}$,a circle of radius 1 about the origin in the plane.$\Rightarrow$the level curve $f^{-1}(1)$

Level Curve: Example

Example 13.3 $f(x, y) = x^2 + y^2$ choose another point (2, 1), $f = 1 \Rightarrow$ the set $\{(x, y) : x^2 + y^2 = 5\} \Rightarrow$the level curve $f^{-1}(5)$
we have drawn the level curves
$f^{-1}(1)$,$f^{-1}(5)$,$f^{-1}(4)$ and $f^{-1}(9)$in Figure 13.7
Every point on the plane lies on one and only one level curve of f. For $z = x^2 + y^2$, all the level curves are circles about the origin except that $f^{-1}(0)$ is just the origin itself.

Functions between Euclidean Spaces

- **Theorem 13.1** Let $f : R^k \to R^1$ be a **linear** function. Then $\exists$ a vector $a \in R^k$ s.t. $f(x) = ax$ for all $x \in R^k$.
- **Theorem 13.2** Let $f : R^k \to R^m$ be a **linear** function. Then $\exists$ an $m \times k$ matrix A s.t. $f(x) = Ax$ for all $x \in R^k$.

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Functions between Euclidean Spaces

Quadratic Forms: A quadratic form on R^k is a real-valued function of the form $Q(x_1, \cdots, x_k) = \sum_{i,j=1}^{k} a_{ij} x_i x_j$ or $Qx_1, \cdots, x_k) = x'Ax$, where A is a unique symmetric matrix. (**Theorem 13.3**)

Functions between Euclidean Spaces

- Monomial: $f : R^k \to R^1$ is a monomial if $f(x_1, \cdots, x_k) = cx_1^{a_1} x_2^{a_2} \cdots x_k^{a_k}$.
- Polynomial : a finite sum of monomial on R^k
- Affine: $f(x) = Ax + b$. $A : m \times k$, b: m-vector

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Monomial: Example

Example 13.6

- $f(x_1, x_2) = -4x_1^2 x_2$ is a monomial of degree three;
- $f(x_1, x_2, x_3) = 3x_1^2 x_2^3 x_3$ is a monomial of degree six;
- a constant function is a monomial of degree zero;
- each term of a linear function is a monomial of degree one
- each term of a quadratic form is a monomial of degree two.

Affine: Example

Example 13.7 If $f : R^k \to R^m$ is polynomial of degree one, then each component of f has the form

$$f_i(x) = a_i \cdot x + b_i.$$

Therefore, f itself has the form: $f(x) = A \cdot x + b$, for some m x k matrix A and some m-vector b. Such function is called an affine function.

Continuous

- Definition: f is continuous at x if $x_n \to x$ then $f(x_n) \to f(x)$.
- **Theorem 13.4** Suppose f and g are continuous at x. Then $f + g$, $f - g$ and $f \cdot g$ are all continuous at x.

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Continuous

- **Theorem 13.5** $f : R^k \to R^m$. f is continuous if and only if $f_i : R^k \to R^1$ is continuous.
- **Theorem 13.7** $f : R^k \to R^m$, $g : R^m \to R^n$. Both f and g are continuous. $g \circ f : R^k \to R^n$ is continuous.

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Continuous: Example

Example 13.8 If f is not continuous at x, there is a sequence $\{x_n\}_{n=1}^{\infty}$ which converges to x and for which $f(x_n)$ does not converge to $f(x)$.

$$f(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases}$$

note that $f(0) = 0$, but $f(x) = 1$ for x arbitrarily close to 0 on the right hand side of 0. The sequence $\{1/n\}$ converges to 0, but $f(1/n) = 1$ converges to 1, which is not $f(0) = 0$. see Figure 13.18.