Mathematical Economics: Lecture 9

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Outline

New Section

Chapter 14: Calculus of Several Variables

Partial Derivatives

Definition: $f : R^n \to R^1$

$$\frac{\partial f}{\partial x_i}(x_1^0, \cdots, x_n^0) =$$

$$\lim_{h \to 0} \frac{f(x_1^0 \cdots x_i^0 + h, \cdots, x_n^0) - f(x_1^0 \cdots, x_i^0 \cdots, x_n^0)}{h}$$

Partial Derivatives: Example

Example 14.1 function $f(x) = 3x^2y^2 + 4xy^3 + 7y$

$$\frac{\partial}{\partial x}(3x^2y^2) = 2x \cdot 3y^2 = 6xy^2,$$
$$\frac{\partial}{\partial x}(4xy^3) = 4y^3,$$
$$\frac{\partial}{\partial x}(7y) = 0.$$

partial derivative of x:

$$\frac{\partial}{\partial x}(3x^2y^2 + 4xy^3 + 7y) = 6xy^2 + 4y^3.$$

Partial Derivatives: Example

$$\frac{\partial}{\partial y}(3x^2y^2) = 3x^2 \cdot 2y = 6x^2y,$$
$$\frac{\partial}{\partial y}(4xy^3) = 3y^2 \cdot 4x = 12xy^2,$$
$$\frac{\partial}{\partial y}(7y) = 0.$$

partial derivative of y:

$$\frac{\partial}{\partial y}(3x^2y^2 + 4xy^3 + 7y) = 6x^2y + 12xy^2.$$

Economics interpretation

$Q = F(K, L)$
Marginal Product of capital (MPK):$(\partial F/\partial K)$
Marginal Product of labor (MPL):$(\partial F/\partial L)$

Economics interpretation

$D = D(P_1, P_2, I)$ demand function for good 1

own Price elasticity of demand: $\frac{P_1}{D}\frac{\partial D(P_1,P_2,I)}{\partial P_1}$

cross price elasticity of demand: $\frac{P_2}{D}\frac{\partial D(P_1,P_2,I)}{\partial P_2}$

income elasticity of demand: $\frac{I\partial D}{D\partial I}$

Total Derivatives

- $F(x^* + \Delta x, y^*) - F(x^*, y^*) \approx \frac{\partial F}{\partial x}(x^*, y^*)\Delta x$
- $F(x^*, y^* + \Delta y) - F(x^*, y^*) \approx \frac{\partial F}{\partial y}(x^*, y^*)\Delta y$
- $F(x^* + \Delta x, y^* + \Delta y) - F(x^*, y^*) \approx \frac{\partial F}{\partial x}(x^*, y^*)\Delta x + \frac{\partial F}{\partial y}(x^*, y^*)\Delta y$

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Total Derivatives

- $dF = \frac{\partial F}{\partial x}(x^*, y^*)dx + \frac{\partial F}{\partial y}(x^*, y^*)dy$
- Example $h = x^3 \ln y$
  $dh = 3x^2 \ln y dx + x^3 \frac{1}{y} dy$

Total Derivatives

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Jacobian

- $dF = \frac{\partial F}{\partial x_1}(x^*)dx_1 + \cdots + \frac{\partial F}{\partial x_n}(x^*)dx_n$
- Jacobian derivative of F at x^*:
  $DF_{x^*} = \left(\frac{\partial F}{\partial x_1}(x^*), \cdots, \frac{\partial F}{\partial x_n}(x^*)\right)$
- $dF = DF_{x^*} dx$, where
  $dx = (dx_1, dx_2, \cdots, dx_n)'$

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Chain Rule

- A curve in R^n: $x(t) = (x_1(t), \cdots, x_n(t))$. $x_i(t)$ are called coordinate function
- **Example 14.5** The line segment connecting (0, 0) and (1, 1) is a curve.One possible parameterization is

$$x(t) = t,\ \text{y(t)=t, } 0 \leq t \leq 1,$$

Another parameterization is

$$x(t) = t^2,\ y(t) = t^2, 0 \leq t \leq 1,$$

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Chain Rule

- Velocity vector (tangent vector): $x'(t) = (x_1'(t), \cdots, x_n'(t))$
- A curve $x(t) = (x_1(t), \cdots, x_n(t))$ is regular if and only if $x_i'(t)$ is continuous and $(x_1'(t), \cdots, x_n'(t)) \neq (0, \cdots, 0)$ for all t.
- Continuously differentiable : $f : R^n \to R^1$, $(\partial f / \partial x_i)$ exists and is continuous for all x_i.

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Chain Rule

- If $X(t) = (x_1(t), \cdots, x_n(t))$ is a C^1 curve on an interval about t_0 and f is a C^1 function on a ball about $X(t_0)$, then $g(t) \equiv f(x_1(t), \cdots, x_n(t))$ is a C^1 function at t_0 and

$$\frac{dg}{dt}(t_0) = \frac{\partial f}{\partial x_1}(X(t_0))x_1'(t_0) + \cdots + \frac{\partial f}{\partial x_n}(X(t_0))x_n'(t_0)$$

Chain Rule:Example 14.7

Example 14.7

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 \\
\textit{let } \text{x(t)=t, } y(t) &= t \\
g(t) &= f(x(t), y(t)) \\
g(t) &= 2t^2 \\
g'(t) &= 4t. \\
g'(1) &= 4
\end{aligned}
$$

Chain Rule: Example 14.7

the same with

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x, \\ \frac{\partial f}{\partial y} &= 2y, \\ g'(1) &= \frac{\partial f}{\partial x}(1,1)\cdot 1 + \frac{\partial f}{\partial y}(1,1)\cdot 1 \\ &= 2\cdot 1 + 2\cdot 1 = 4.\end{aligned}$$

Directional Derivative and Gradients

$$X = X^* + tV$$
$$g(t) = F(X * +tV) = F(x_1^* + tv_1, \cdots, x_n^* + tv_n)$$
$$g'(0) = \frac{\partial F}{\partial x_1}(X^*)v_1 + \cdots + \frac{\partial F}{\partial x_n}(X^*)v_n$$

$\left(\frac{\partial F}{\partial x_1}(X^*), \cdots, \frac{\partial F}{\partial x_n}(X^*)\right) \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = DF_{x^*}V$ the derivative of F at x^* in the direction of V.

Directional Derivative and Gradients

Gradient: $\nabla F_{x^*} = \begin{pmatrix} \frac{\partial F}{\partial x_1}(X^*) \\ \vdots \\ \frac{\partial F}{\partial x_n}(X^*) \end{pmatrix}$

Theorem

Theorem 14.2 $\nabla F(x)$ points at x into the direction in which F increases most rapidly.

Theorem

Example 14.11 Consider again the production function:

$$Q = F(K, L) = 4K^{3/4}L^{1/4}.$$

Current input bundles is $(10000, 625)$. If we want to know in what proportions we should add K and L to increase production most rapidly, we compute the gradient vector

$$\nabla F(10,000, 625) = \begin{pmatrix} 1.5 \\ 8 \end{pmatrix}$$

Explicit Function

- $F = (f_1, \cdots, f_m) : R^n \to R^m$
- $f_1(x^* + \Delta x) - f_1(x^*) \approx \frac{\partial f_1}{\partial x_1}(x^*)\Delta x_1 + \cdots + \frac{\partial f_1}{\partial x_n}(x^*)\Delta x_n$
- $f_2(x^* + \Delta x) - f_2(x^*) \approx \frac{\partial f_2}{\partial x_1}(x^*)\Delta x_1 + \cdots + \frac{\partial f_2}{\partial x_n}(x^*)\Delta x_n$
- $f_m(x^* + \Delta x) - f_m(x^*) \approx \frac{\partial f_m}{\partial x_1}(x^*)\Delta x_1 + \cdots + \frac{\partial f_m}{\partial x_n}(x^*)\Delta x_n$

Explicit Function

$$F(X^* + \Delta X) - F(X^*) \approx$$
$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X^*) & \cdots & \frac{\partial f_1}{\partial x_n}(X^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(X^*) & \cdots & \frac{\partial f_m}{\partial x_n}(X^*) \end{pmatrix} \begin{pmatrix} \Delta x_1 \\ \vdots \\ \Delta x_n \end{pmatrix}$$

Jacobian Derivative

- Jacobian Derivative of F at X^*

$$DF_{x^*} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(X^*) & \cdots & \frac{\partial f_1}{\partial x_n}(X^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1}(X^*) & \cdots & \frac{\partial f_m}{\partial x_n}(X^*) \end{pmatrix}$$

- Example 14.13

Theorems

Theorem 14.3 $F : R^n \to R^m$ and $a : R^1 \to R^n$, $g(t) = F(a(t))$, then $g_i'(t) = Df_i(a(t))a'(t)$ and $g'(t) = DF(a(t))a'(t)$

Theorems

Theorem 14.4 $F : R^n \to R^m$ and $A : R^s \to R^n$, $H = F \circ A$, then $DH(s^*) = DF(x^*)DA(s^*)$

Theorems

Theorem 14.5 $\frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$

Higher Order derivatives & Hessian

C^1 : continuously differentiable, f' is continuous
C^2 : twice continuously differentiable, f'' is continuous
$f : R^n \to R^1$

- Hessian or Hessian Matrix:

$$D^2 F_{x^*} = \begin{pmatrix} \frac{\partial^2 f}{\partial (x_1)^2}(X^*) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(X^*) \\ \vdots & \vdots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(X^*) & \cdots & \frac{\partial^2 f}{\partial (x_n)^2}(X^*) \end{pmatrix}$$

Example 14.15

Example 14.15

$$Q = 4K^{3/4}L^{1/4}.$$

$$\frac{\partial Q}{\partial K} = 3K^{-1/4}L^{1/4}, \ \frac{\partial Q}{\partial L} = K^{3/4}L^{-3/4}$$

Example 14.15

then

$$\frac{\partial^2 Q}{\partial L \partial K} = \frac{\partial Q}{\partial L}(\frac{\partial Q}{\partial K}) = \frac{\partial}{\partial L}(3K^{-1/4}L^{1/4}) = \frac{3}{4}K^{-1/4}L^{-3/4}$$

$$\frac{\partial^2 Q}{\partial K \partial L} = \frac{\partial Q}{\partial K}(\frac{\partial Q}{\partial L}) = \frac{\partial}{\partial K}(K^{3/4}L^{-3/4}) = \frac{3}{4}K^{-1/4}L^{-3/4}$$

$$\frac{\partial^2 Q}{\partial L^2} = \frac{\partial Q}{\partial L}(\frac{\partial Q}{\partial L}) = \frac{\partial}{\partial L}(K^{3/4}L^{-3/4}) = -\frac{3}{4}K^{3/4}L^{-7/4}$$

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